MANCHESTER 1824
The University of Manchester

Combinatorial Mesh Calculus (CMC): Lecture 5

Lectured by: Dr. Kiprian Berbatov[1]

Lecture Notes Compiled by: Muhammad Azeem[1]

Under the supervision of: Prof. Andrey P. Jivkov[1]

[1] Department of Mechanical and Aerospace Engineering, The University of Manchester, Oxford Road, Manchester M13 9PL, UK

Dimension

Let R be a commutative ring with unity (CRWU), and let V be an R–module.

Definitions

- V is **free** if it has a basis, i.e. a subset $E \subseteq V$ such that every $v \in V$ can be written uniquely as a finite R–linear combination of elements of E.
- V is **finite–dimensional** if it has a finite basis.
- If $E = \{e_1, \ldots, e_n\}$ is a basis of V, we write $\dim_E V = n$.

Remarks: Free modules generalize vector spaces when the underlying field is replaced by a ring. The existence of a basis ensures that every element can be expressed uniquely in terms of simple building blocks.

Well–defined dimension for free modules

Theorem

Let R be a CRWU and V a finite-dimensional R–module. If E and F are bases of V, then

$$\dim_E V = \dim_F V.$$

We therefore denote the unique value by $\dim V$ and call it the *rank* of V.

Proof It suffices to show: if $R^m \cong R^n$ as R–modules, then $m = n$. Indeed, a finite-dimensional module V with a basis of size m is isomorphic to R^m, so any two bases yield $R^m \cong R^n$.
Let $\mathfrak{m}$ be a maximal ideal of R; then $k := R/\mathfrak{m}$ is a field. Tensoring the isomorphism $R^m \cong R^n$ with k over R gives

$$k \otimes_R R^m \;\cong\; k \otimes_R R^n.$$

Proof (Continue)

Since $k \otimes_R R \cong k$ and tensor commutes with finite direct sums,

$$k \otimes_R R^m \cong k^m, \qquad k \otimes_R R^n \cong k^n$$

as k–vector spaces. Hence $k^m \cong k^n$ as vector spaces, so $m = n$. Therefore, any two bases of a finite-dimensional R–module have the same cardinality.

Interpretation: If a module has two different bases, they must contain the same number of elements. This number - the dimension or *rank*-measures the intrinsic "size" of the module, just as dimension does for vector spaces.

Examples of Dimension

Let R be a CRWU, $m, n \in \mathbb{N}$.

1 dim $R^n = n$, because each element $(a_1, \ldots, a_n)$ can be written uniquely as

$$(a_1, \ldots, a_n) = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n,$$

where $e_i = (0, \ldots, \underbrace{1}_{i}, \ldots, 0)$ are linearly independent and span R^n.

2 dim $M_{m\times n}(R) = mn$, because every matrix $A = (a_{ij})$ can be expressed uniquely as $A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij}$, where E_{ij} has 1 in position (i, j) and 0 elsewhere. These mn matrices are linearly independent and span $M_{m\times n}(R)$.

Examples of Dimension

Let R be a CRWU, $m, n \in \mathbb{N}$.

3 dim $\mathbf{0} = 0$, because the zero module $\{0\}$ contains only the zero element. The empty set $\varnothing$ is vacuously linearly independent and spans $\{0\}$, so it serves as a basis.

4 dim $R[x] = \infty$, because the set $\{1, x, x^2, x^3, \dots\}$ is linearly independent and spans all polynomials. Every polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a finite R–linear combination of these monomials.

Span and Equivalent Descriptions

Definition:

Let R be a CRWU, V an R–module, and $v_1, \dots, v_m \in V$.

$$\text{span}\{v_1, \dots, v_m\} := \left\{ \sum_{i=1}^{m} \lambda_i v_i \;\middle|\; \lambda_1, \dots, \lambda_m \in R \right\}$$

$$= \{\, w \in V \mid \exists \lambda_1, \dots, \lambda_m \in R,\; w = \sum_{i=1}^{m} \lambda_i v_i \,\}.$$

Remark 1 (Matrix image equals span of columns)

If $A \in M_{m \times n}(R)$ has column vectors $c_1, \dots, c_n \in R^m$, then

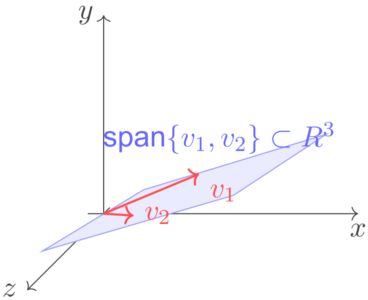
$$\text{Im}(A) = \{Av \mid v \in R^n\} = \text{span}\{c_1, \dots, c_n\} \subseteq R^m.$$

Span

Remark 2 (Span is a submodule)

span$\{v_1, \ldots, v_m\}$ is a submodule of V.

Diagrammatic View: A Span as a Submodule (Plane in $\mathbb{R}^3$)



Proof of the Remark

Proof of Remark 1.

Write $v = (\lambda_1, \ldots, \lambda_n)^\top \in R^n$. Then

$$Av = \lambda_1 c_1 + \cdots + \lambda_n c_n \in \text{span}\{c_1, \ldots, c_n\}.$$

Thus $\text{Im}(A) \subseteq \text{span}\{c_j\}$. Conversely, any linear combination $\sum \lambda_j c_j$ equals $A(\lambda_1, \ldots, \lambda_n)^\top$, hence belongs to $\text{Im}(A)$. Therefore, equality holds. $\square$

Proof of the Remark

Proof of Remark 2.

Let $W := \text{span}\{v_1, \dots, v_m\}$. If $w = \sum \lambda_i v_i$ and $w' = \sum \mu_i v_i$, then for any $a, b \in R$,

$$aw + bw' = \sum (a\lambda_i + b\mu_i) v_i \in W,$$

so W is closed under linear combinations, hence under addition and R–scalar multiplication; thus W is a submodule of V. $\square$

Example of a Span

Polynomial Module

Let $S = \{1, x, x^2\} \subseteq \mathbb{Q}[x]$. Then

$$\text{span}_{\mathbb{Q}}(S) = \left\{\, a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{Q} \,\right\},$$

which is the $\mathbb{Q}$–vector space (or $\mathbb{Q}$–module) of all quadratic polynomials.

Reasoning.

- Every quadratic polynomial $p(x) = a_0 + a_1 x + a_2 x^2$ can be uniquely written as a linear combination of $1, x, x^2$.
- The set $\{1, x, x^2\}$ is linearly independent, since $c_0 \cdot 1 + c_1 x + c_2 x^2 = 0 \Rightarrow c_0 = c_1 = c_2 = 0$.
- Therefore, $\{1, x, x^2\}$ forms a **basis** of $\text{span}_{\mathbb{Q}}(S)$.

Interpretation. Geometrically, the span operation constructs the smallest subspace (submodule) of $\mathbb{Q}[x]$ that contains S.

Definition and Notation

Let R be a CRWU, V, W be R–modules, and $f : V \to W$ a function.

Definition (Linear map / R–module homomorphism)

f is **linear** if for all $v_1, v_2 \in V$ and $\lambda \in R$,

$$\begin{aligned} f(v_1 + v_2) &= f(v_1) + f(v_2), \\ f(\lambda v) &= \lambda f(v). \end{aligned}$$

Notation

The set of all R–linear maps $V \to W$ is denoted

$$\mathrm{Hom}_R(V, W) \quad \text{or} \quad \mathrm{Hom}(V, W) \quad \text{or} \, \mathcal{L}(V, W).$$

Examples of Linear Maps

Let R be a CRWU, V, W be R–modules.

1. **Zero map** $0 : V \to W$, $0(v) = 0$ is linear (both axioms hold trivially).
2. **Identity** $\mathrm{id}_V : V \to V$ is linear.
3. **Matrix map** If $V = R^n$, $W = R^m$ and $A \in M_{m\times n}(R)$, define $\mathcal{A}(v) = Av$. Then

   $$\mathcal{A}(v+w) = A(v+w) = Av+Aw, \qquad \mathcal{A}(\lambda v) = A(\lambda v) = \lambda(Av),$$

   so $\mathcal{A}$ is linear.
4. **Integration** on $C^0[0,1]$: Let $V = W = C^0[0,1]$ over $R = \mathbb{R}$ and

   $$(If)(x) = \int_0^x f(t)\,dt.$$

   Then I is linear by linearity of the integral:

   $$I(f+g) = If + Ig, \qquad I(\lambda f) = \lambda(If).$$

MANCHESTER 1824
The University of Manchester

Kernel - Image; Submodule Properties

Definitions

Let $f \in \mathrm{Hom}_R(V, W)$.

$$\ker f := \{\, v \in V \mid f(v) = 0_W \,\},$$
$$\operatorname{Im} f := \{\, f(v) \mid v \in V \,\} = \{\, w \in W \mid \exists v \in V,\ w = f(v) \,\}.$$

Proposition

Let $f \in \mathrm{Hom}_R(V, W)$. Then $\ker f$ is a submodule of V, and $\operatorname{Im} f$ is a submodule of W.

Kernel - Image; Submodule Properties

Proof

If $u, v \in \ker f$ and $\lambda, \mu \in R$, then

$$f(\lambda u + \mu v) = \lambda f(u) + \mu f(v) = 0,$$

so $\lambda u + \mu v \in \ker f$. Thus $\ker f \leq V$.
If $y_1 = f(v_1), y_2 = f(v_2)$ and $\lambda, \mu \in R$, then

$$\lambda y_1 + \mu y_2 = \lambda f(v_1) + \mu f(v_2) = f(\lambda v_1 + \mu v_2) \in \operatorname{Im} f.$$

Hence $\operatorname{Im} f \leq W$.

Example

Let $R = \mathbb{R}$ and

$$V = W = \{ a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \},$$

define $D : V \to V$ by $Df = f'$.

Computations

$$D(2 + 4x + x^2) = 4 + 2x.$$

$$\ker D = \{ a_0 \mid a_0 \in \mathbb{R} \} \quad \text{(constant polynomials)},$$

$$\operatorname{Im} D = \{ a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R} \} \quad \text{(linear polynomials)}.$$

Justification.

$Df = 0$ iff f is constant, hence $\ker D$ are the constants. If $f = a_0 + a_1 x + a_2 x^2$, then $Df = a_1 + 2a_2 x$ is any linear polynomial: given $\alpha + \beta x$ choose $a_1 = \alpha$, $a_2 = \beta/2$. Thus the image equals the set of linear polynomials. $\square$

Matrix Image as Span of Columns

$$\begin{tikzcd}
& \mathbb{R}^n \arrow[r, "\mathcal{A}"] \arrow[dl, dashed, "\text{choose } v = (\lambda_1, \ldots, \lambda_n)^\top"'] & \mathbb{R}^m \\
\text{coefficients } (\lambda_i) \arrow[r, mapsto] & \lambda_1 c_1 + \cdots + \lambda_n c_n \in \text{span}\{c_1, \ldots, c_n\} \arrow[ur] &
\end{tikzcd}$$

$$\text{Im}(\mathcal{A}) = \text{span}\{c_1, \ldots, c_n\} \subseteq R^m.$$

Proposition: f injective $\iff$ $\ker f = \{0\}$

Proposition.

Let R be a CRWU, V, W be R–modules, and $f \in \mathrm{Hom}_R(V, W)$. Then

$$f \text{ is injective} \iff \ker f = \{0\}.$$

Proof.

($\Rightarrow$) If f is injective and $v \in \ker f$, then $f(v) = 0 = f(0)$, hence $v = 0$. So $\ker f = \{0\}$.
($\Leftarrow$) Suppose $\ker f = \{0\}$ and $f(v_1) = f(v_2)$. Then $f(v_1 - v_2) = 0$, so $v_1 - v_2 \in \ker f$, hence $v_1 - v_2 = 0$, i.e. $v_1 = v_2$. Thus f is injective. $\square$

Definition and Equivalent Characterization

The University of Manchester

Definition (Isomorphism).

Let R be a CRWU, V, W R–modules, and $f \in \mathrm{Hom}_R(V, W)$. We say f is an *isomorphism* if there exists $g \in \mathrm{Hom}_R(W, V)$ such that

$$g \circ f = \mathrm{id}_V, \qquad f \circ g = \mathrm{id}_W.$$

Equivalently

f is bijective. In this case V and W are *isomorphic*, written $V \cong W$.

Flattening Matrices is an Isomorphism

Let R be a CRWU and $m, n \in \mathbb{N}$. Put $V = M_{m\times n}(R)$ and $W = R^{mn}$. Define

$$F : V \to W, \quad F\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\right) = \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \\ a_{21} \\ \vdots \\ a_{2n} \\ \vdots \\ a_{m1} \\ \vdots \\ a_{mn} \end{pmatrix} \in R^{mn}.$$

Flattening Matrices is an Isomorphism

Claim:

F is a linear isomorphism.

Linearity:

$F(A+B) = F(A) + F(B)$ and $F(\lambda A) = \lambda F(A)$ hold entrywise.

Injective:

$F(A) = 0$ implies all $a_{ij} = 0$, hence $A = 0$.

Surjective:

Given $(b_1, \ldots, b_{mn})^\top \in R^{mn}$, place entries row-by-row into a matrix B; then $F(B) = (b_1, \ldots, b_{mn})^\top$.
Therefore F is a linear bijection, i.e. an isomorphism $M_{m\times n}(R) \cong R^{mn}$.

The University of Manchester

Universal Property of Free Modules

Proposition (Extension from a basis).

Let R be a CRWU, V a free R–module with basis $S = \{e_1, \ldots, e_n\}$ (possibly infinite index set), W an R–module, and pick arbitrary $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $f \in \mathrm{Hom}_R(V, W)$ such that

$$f(e_i) = w_i, \quad i = 1, \ldots, n.$$

Proof

Proof.

Every $v \in V$ can be written uniquely as $v = \sum_{i=1}^{n} \lambda_i e_i$ with $\lambda_i \in R$ (finite sum if S infinite). Define $f(v) := \sum_{i=1}^{n} \lambda_i w_i$. This is well-defined by uniqueness of the coefficients. For linearity: if $v = \sum \lambda_i e_i$ and $u = \sum \mu_i e_i$, then

$$f(v+u) = \sum (\lambda_i + \mu_i) w_i = \sum \lambda_i w_i + \sum \mu_i w_i = f(v) + f(u),$$

and $f(\alpha v) = \sum (\alpha \lambda_i) w_i = \alpha \sum \lambda_i w_i = \alpha f(v)$. By construction $f(e_i) = w_i$.
Uniqueness: if g is another linear map with $g(e_i) = w_i$, then for any $v = \sum \lambda_i e_i$, $g(v) = \sum \lambda_i g(e_i) = \sum \lambda_i w_i = f(v)$. Hence $g = f$. $\square$

Example

$R^n \cong \text{Free}_R(S)$

Let R be a CRWU, $n \in \mathbb{N}$, and $S = \{e_1, \ldots, e_n\}$ a finite set. Define

$$f : R^n \to \text{Free}_R(S), \qquad f(1, 0, \ldots, 0) = e_1, \ \ldots, \ f(0, \ldots, 0, 1) = e_n,$$

and extend R–linearly. Then for $(\lambda_1, \ldots, \lambda_n) \in R^n$,

$$f(\lambda_1, \ldots, \lambda_n) = \lambda_1 e_1 + \cdots + \lambda_n e_n.$$

By the previous proposition, f is linear and bijective (its inverse sends $\sum \lambda_i e_i$ to $(\lambda_1, \ldots, \lambda_n)$). Hence $R^n \cong \text{Free}_R(S)$.

Coordinate Map with Respect to a Basis

Let R be a CRWU and V a finite-dimensional R–module with ordered basis $e = (e_1, \ldots, e_n)$.

Definition (Coordinates). For $v \in V$ written uniquely as $v = \lambda_1 e_1 + \cdots + \lambda_n e_n$, define

$$(\cdot)^e : V \to R^n, \qquad v^e := (\lambda_1, \ldots, \lambda_n)^\top.$$

Proposition. $(\cdot)^e$ is a linear isomorphism.

Proof.

Linearity is immediate from linearity of coordinate extraction.
Injectivity: $v^e = 0$ implies all coordinates $= 0$, hence $v = 0$.
Surjectivity: given $(\mu_1, \ldots, \mu_n)^\top$, take $v = \sum \mu_i e_i$, then $v^e = (\mu_1, \ldots, \mu_n)^\top$. $\square$

Example

Setup. Let $V = \{ a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in R \}$ be the R–module (or vector space, if R is a field) of quadratic polynomials, and let $e = (1, x, x^2)$ be its ordered basis.

Coordinate Map

Every element $v \in V$ can be uniquely written as

$$v = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2,$$

so its coordinate vector with respect to e is

$$v^e = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \in R^3.$$

Example

Interpretation.

- The coordinate map $(\cdot)^e : V \to R^3$ translates each polynomial into its list of coefficients.
- It is a linear isomorphism — addition and scalar multiplication of polynomials correspond exactly to those of their coordinate vectors:

$$(p+q)^e = p^e + q^e, \qquad (\lambda p)^e = \lambda p^e.$$

- Hence V is *algebraically identical* to R^3 under this basis, just viewed in a different "language" — coefficients instead of components.

Visualization. Think of the polynomial $a_0 + a_1 x + a_2 x^2$ as a point in R^3 whose coordinates are (a_0, a_1, a_2) — the space of all quadratic shapes parameterized by their coefficients.

Hom is a Submodule of W^V

Definition

Let W^V denote all functions $V \to W$ with pointwise operations:

$$(f+g)(v) := f(v) + g(v),$$
$$(\lambda f)(v) := \lambda f(v).$$

Proposition.

$\mathrm{Hom}_R(V, W) \subseteq W^V$ is a submodule (closed under $+$ and R–scalars).

Hom is a Submodule of W^V

Proof.

If $f, g \in \mathrm{Hom}_R(V, W)$ and $\lambda \in R$, for any $v, u \in V$:

$$\begin{aligned}(f+g)(v+u) =& f(v+u) + g(v+u) = f(v) + f(u) + g(v) + g(u) \\ =& (f+g)(v) + (f+g)(u),\end{aligned}$$

$$(f+g)(\alpha v) = f(\alpha v) + g(\alpha v) = \alpha f(v) + \alpha g(v) = \alpha(f+g)(v).$$

So $f+g$ is linear.
Similarly $(\lambda f)(v+u) = \lambda f(v+u) = \lambda f(v) + \lambda f(u)$ and $(\lambda f)(\alpha v) = \lambda\alpha f(v) = \alpha(\lambda f)(v)$, hence λf is linear.
Therefore $\mathrm{Hom}_R(V, W)$ is an R–submodule of W^V. $\square$

Example in $\mathbb{R}$

$h = 2D - 3\,\mathrm{id}$ on Quadratics

Let $R = \mathbb{R}$ and $V = \{a_0 + a_1 x + a_2 x^2\}$ with basis $e = (1, x, x^2)$. Put

$$f = D, \quad g = \mathrm{id}_V, \quad h = 2f - 3g.$$

Then for $p(x) = a_0 + a_1 x + a_2 x^2$,

$$\begin{aligned} h(p) =& 2(a_1 + 2a_2 x) - 3(a_0 + a_1 x + a_2 x^2) \\ =& (-3a_0 + 2a_1) \; + \; (-3a_1 + 4a_2)x \; + \; (-3a_2)x^2. \end{aligned}$$

In coordinates $p^e = (a_0, a_1, a_2)^\top$,

$$h^e(p) = \begin{pmatrix} -3a_0 + 2a_1 \\ -3a_1 + 4a_2 \\ -3a_2 \end{pmatrix}.$$

Matrix of a Linear Map

Let V, W be finite-dimensional R–modules, with ordered bases

$$e = (e_1, \ldots, e_n) \text{ for } V, \qquad f = (f_1, \ldots, f_m) \text{ for } W.$$

For $\mathcal{A} \in \mathrm{Hom}_R(V, W)$, write for each $1 \le j \le n$:

$$\mathcal{A}(e_j) = a_{1j} f_1 + \cdots + a_{mj} f_m.$$

Definition. The matrix of $\mathcal{A}$ w.r.t. (e, f) is

$$(\mathcal{A})_e^f = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m \times n}(R).$$

Example

Matrix of $h = 2D - 3\,\mathrm{id}$

With $e = f = (1, x, x^2)$ and the computations

$$h(1) = -3, \qquad h(x) = 2 - 3x, \qquad h(x^2) = 4x - 3x^2,$$

their coordinate columns (w.r.t. e) are

$$[-3, 0, 0]^\top, \quad [2, -3, 0]^\top, \quad [0, 4, -3]^\top.$$

Hence

$$(h)_e^f = \begin{bmatrix} -3 & 2 & 0 \\ 0 & -3 & 4 \\ 0 & 0 & -3 \end{bmatrix}.$$

Composition Corresponds to Matrix Product

Let U, V, W be finite-dimensional R–modules with ordered bases

$$e = (e_1, \dots, e_p), \quad f = (f_1, \dots, f_m), \quad g = (g_1, \dots, g_n).$$

Let $A \in \mathrm{Hom}_R(U, V)$ and $B \in \mathrm{Hom}_R(V, W)$. Then $B \circ A \in \mathrm{Hom}_R(U, W)$ and

$$(A)_e^f \in M_{m \times p}(R), \quad (B)_f^g \in M_{n \times m}(R), \quad (B \circ A)_e^g \in M_{n \times p}(R),$$

with the identity

$$(B \circ A)_e^g = (B)_f^g \, (A)_e^f .$$

$$\begin{tikzcd} U \arrow[r, "A"] \arrow[rr, bend right, "B \circ A"'] & V \arrow[r, "B"] & W \end{tikzcd}$$

Proof

Proof.

For each j, write $A(e_j) = \sum_{i=1}^{m} \alpha_{ij} f_i$ and $B(f_i) = \sum_{k=1}^{n} \beta_{ki} g_k$. Then

$$(B \circ A)(e_j) = B\Big(\sum_i \alpha_{ij} f_i\Big) = \sum_i \alpha_{ij} \Big(\sum_k \beta_{ki} g_k\Big)$$
$$= \sum_k \Big(\sum_i \beta_{ki} \alpha_{ij}\Big) g_k.$$

Thus the (k, j)–entry of $(B \circ A)_e^g$ is $\sum_i \beta_{ki}\alpha_{ij}$, i.e. the matrix product $(B)_f^g (A)_e^f$. $\square$

MANCHESTER 1824
The University of Manchester

Summary

- Defined free and finite–dimensional R–modules; proved dimension is well–defined for finite-dimensional modules (IBN via reduction mod maximal ideals).
- Computed dimensions of R^n, $M_{m\times n}(R)$, $\mathbf{0}$, and $R[x]$.
- Defined span; proved image of a matrix equals the span of its columns; proved span is a submodule.
- Defined linear maps; verified linearity in key examples (zero, identity, matrix, integral).
- Defined kernel and image; proved they are submodules; computed them for D on quadratics.

MANCHESTER 1824
The University of Manchester

Thanks