Combinatorial Mesh Calculus (CMC): Lecture 6

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Proposition

$\dim \mathrm{Hom}_R(V, W) = (\dim V)(\dim W)$

Let R be a commutative ring with unity (CRWU). Let V, W be finite-dimensional R–modules with

$$\dim V = n, \qquad \dim W = m.$$

$$\dim \mathrm{Hom}_R(V, W) = mn.$$

Proof

Choose ordered bases $e = (e_1, \ldots, e_n)$ of V and $f = (f_1, \ldots, f_m)$ of W. For each $A \in \mathrm{Hom}_R(V, W)$ write

$$A(e_j) = \sum_{i=1}^{m} a_{ij} f_i \quad (1 \leq j \leq n).$$

Proposition

Proof.

This identifies $A \longleftrightarrow (a_{ij}) \in M_{m\times n}(R)$. The correspondence

$$\Phi : \mathrm{Hom}_R(V, W) \simeq M_{m\times n}(R), \quad A \mapsto (a_{ij})$$

is an R–module isomorphism (linearity is entrywise; bijectivity follows by defining a map from any matrix to the unique A with those columns). Hence

$$\mathrm{Hom}_R(V, W) \cong R^{mn} \quad \Rightarrow \quad \dim \mathrm{Hom}_R(V, W) = mn. \qquad \square$$

Dual Module

Definition.

For a CRWU R and an R–module V, the *dual module* is

$$V^* := \text{Hom}_R(V, R)\ [= \{f : V \to R | f \text{ is linear}\}]\,.$$

Example

$$\begin{aligned}(R^2)^* =& \{f : R^2 \to R \text{ linear}\} \\ =& \{\, f(x_1, x_2) = \lambda_1 x_1 + \lambda_2 x_2 \mid \lambda_1, \lambda_2 \in R\,\}.\end{aligned}$$

Finite-dimensional Case

Remark: Dimension and Basis Dependence

Let R be a commutative ring with unity (CRWU), and let V be a finite–dimensional (free) R–module. Then

$$\text{dim}(V^*) = \text{dim}\left(\text{Hom}_R(V, R)\right) = \text{dim}(V).$$

Indeed, each R–linear map $f : V \to R$ is determined uniquely by its values on a basis of V. If V has basis $\{e_1, \ldots, e_n\}$, the dual module V^* has the dual basis $\{e^1, \ldots, e^n\}$ where $e^i(e_j) = \delta^i_j$; hence both spaces have the same number of basis elements.

Summary. Although $\text{dim}\, V = \text{dim}\, V^*$ and we can identify them under a fixed basis, such identification is artificial - only the double dual V^{**} can be identified with V *canonically*.

Finite-dimensional Case

Interpretation.

- There exists an isomorphism $V \cong V^*$, but it is *not canonical*: it depends on the chosen basis. Different choices of basis lead to different identifications between V and its dual.
- Consequently, in general algebraic treatments (over rings that are not fields), we always distinguish V from V^*.
- In the language of physics and differential geometry:
  - Elements of V are *contravariant vectors* — typically represented as column vectors.
  - Elements of V^* are *covariant vectors* — typically represented as row vectors or linear functionals acting on V.
- Thus, V and V^* play dual but complementary roles: one represents directions, the other represents measurements (functionals) on those directions.

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Dual Basis

Kronecker Delta

Let V be finite-dimensional with ordered basis $e = (e_1, \dots, e_n)$. The *dual basis* $e^* = (e^1, \dots, e^n)$ is the family of linear forms $e^i : V \to R$ defined by

$$e^i(e_j) = \delta^i_j \quad (1 \le i, j \le n),$$

where the *Kronecker delta* is

$$\delta^i_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Example (standard $\mathbb{R}^2$). With $e_1 = (1, 0)$, $e_2 = (0, 1)$, the dual basis is given by projections

$$e^1(x_1, x_2) = x_1, \qquad e^2(x_1, x_2) = x_2.$$

Reconstruction via Dual Basis

Proposition

Let V be finite-dimensional, $e = (e_1, \dots, e_n)$ a basis and $e^* = (e^1, \dots, e^n)$ its dual. Then for every $v \in V$,

$$v = \sum_{i=1}^{n} e^i(v)\, e_i.$$

Proof.

Write $v = \sum_{j=1}^{n} \lambda_j e_j$. Apply e^i:

$$e^i(v) = \sum_{j=1}^{n} \lambda_j e^i(e_j) = \sum_{j=1}^{n} \lambda_j \delta_j^i = \lambda_i.$$

Hence $\sum_i e^i(v) e_i = \sum_i \lambda_i e_i = v$. $\square$

Standard & Nonstandard Basis in $\mathbb{R}^2$

Example (standard $\mathbb{R}^2$).

For $v = (v_1, v_2)$ and $e = (e_1, e_2)$ standard,

$$v = e^1(v)e_1 + e^2(v)e_2 = v_1(1,0) + v_2(0,1) = (v_1, v_2).$$

Example (nonstandard $\mathbb{R}^2$).

Let $e_1 = (1, 2)$, $e_2 = (4, 1)$. Seek $e^1, e^2 \in (\mathbb{R}^2)^*$ such that

$$e^1(e_1) = 1,\ e^1(e_2) = 0, \qquad e^2(e_1) = 0,\ e^2(e_2) = 1.$$

Example

Solve for e^1. Write $e^1(x_1, x_2) = \lambda_1 x_1 + \lambda_2 x_2$. Then

$$\lambda_1 + 2\lambda_2 = 1, \qquad 4\lambda_1 + \lambda_2 = 0.$$

From the second, $\lambda_2 = -4\lambda_1$. Substitute in the first:

$$\lambda_1 - 8\lambda_1 = 1 \Rightarrow -7\lambda_1 = 1 \Rightarrow \lambda_1 = -\frac{1}{7}, \quad \lambda_2 = \frac{4}{7}.$$

Thus $e^1(x_1, x_2) = -\frac{1}{7}x_1 + \frac{4}{7}x_2$.
Solve for e^2. Let $e^2(x_1, x_2) = \mu_1 x_1 + \mu_2 x_2$. Then

$$\mu_1 + 2\mu_2 = 0, \qquad 4\mu_1 + \mu_2 = 1.$$

Example

From the first, $\mu_1 = -2\mu_2$. Substitute:

$$-8\mu_2 + \mu_2 = 1 \Rightarrow -7\mu_2 = 1 \Rightarrow \mu_2 = -\frac{1}{7}, \quad \mu_1 = \frac{2}{7}.$$

Thus $e^2(x_1, x_2) = \frac{2}{7}x_1 - \frac{1}{7}x_2$.
Check. One verifies $e^i(e_j) = \delta^i_j$ and the reconstruction

$$v = e^1(v)\, e_1 + e^2(v)\, e_2 \quad \text{for all } v \in \mathbb{R}^2.$$

Matrix View

Dual Rows are the Inverse Matrix

Let E be the 2×2 matrix with columns e_1, e_2:

$$E = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}, \qquad \det E = 1 \cdot 1 - 4 \cdot 2 = -7 \neq 0.$$

Then

$$E^{-1} = \frac{1}{-7} \begin{bmatrix} 1 & -4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} & \frac{4}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{bmatrix}.$$

Observation. The rows of E^{-1} are exactly the coefficient vectors of e^1 and e^2:

$$e^1(x_1, x_2) = \begin{bmatrix} -\frac{1}{7} & \frac{4}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad e^2(x_1, x_2) = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Thus (e^1, e^2) corresponds to E^{-1} and $(\cdot)$-coordinates satisfy $v^e = E^{-1} v^{\text{std}}$.

Definition

Dual Map

Let V, W be R–modules and $\phi \in \mathrm{Hom}_R(V, W)$. The *dual map*

$$\phi^* : W^* \longrightarrow V^*, \qquad \phi^*(g) := g \circ \phi$$

is R–linear. In evaluation form, for $g \in W^*$ and $v \in V$,

$$(\phi^* g)(v) = g\left(\phi(v)\right).$$

$$\begin{tikzcd}
V \arrow[rr, "\phi"] \arrow[d, "{(\phi^* g)(\cdot) = g(\phi(\cdot))}"'] & & W \arrow[d, "g"] \\
R \arrow[rr, dash, "="] & & R
\end{tikzcd}$$

Corollary: Coordinates via Dual Basis

Let R be a CRWU, V, W finite-dimensional R–modules with ordered bases

$$e = (e_1, \ldots, e_n) \text{ of } V, \qquad f = (f_1, \ldots, f_m) \text{ of } W,$$

and dual bases $e^* = (e^1, \ldots, e^n)$, $f^* = (f^1, \ldots, f^m)$. For $\phi \in \mathrm{Hom}_R(V, W)$ and each j,

$$\phi(e_j) = \sum_{i=1}^{m} f^i(\phi(e_j))\, f_i.$$

Hence the matrix of ϕ w.r.t. (e, f) is

$$(\phi)_e^f = \{f^i(\phi(e_j))\}_{1\leq j\leq n}^{1\leq i\leq m} \in M_{m\times n}(R).$$

Proof.

By definition of the dual basis, any $w \in W$ decomposes as $w = \sum_i f^i(w) f_i$. Apply this to $w = \phi(e_j)$. $\square$

Proposition

Matrix of ϕ^* is the Transpose of $(\phi)^f_e$

Let V, W be finite-dimensional, $e = (e_1, \ldots, e_n)$, $f = (f_1, \ldots, f_m)$ with dual bases $e^* = (e^1, \ldots, e^n)$, $f^* = (f^1, \ldots, f^m)$. For $\phi \in \mathrm{Hom}_R(V, W)$,

$$(\phi^*)^{e^*}_{f^*} = \left((\phi)^f_e\right)^T .$$

Proof

Write $\phi(e_j) = \sum_{i=1}^m a_{ij} f_i$; thus $(\phi)^f_e = (a_{ij})$. We compute the coordinates of $\phi^*(f^i) \in V^*$ in the basis e^*:

$$\phi^*(f^i) = f^i \circ \phi \in V^*,$$

so for each j, $e^j\left(\phi^*(f^i)\right) = \left(\phi^*(f^i)\right)(e_j) = f^i(\phi(e_j))$. But $f^i(\phi(e_j)) = a_{ij}$ by the very definition of the coefficients a_{ij}.

Proof

Hence the j-th coordinate of $\phi^*(f^i)$ w.r.t. e^* equals a_{ij}.
Therefore, the matrix with columns $\left[\phi^*(f^1)\right]_{e^*}, \ldots, \left[\phi^*(f^m)\right]_{e^*}$ is (a_{ij}) with indices swapped, i.e.

$$(\phi^*)^{e^*}_{f^*} = (a_{ji}) = \left((\phi)^f_e\right)^T .$$

$$\begin{tikzcd}
V \arrow[r, "\phi"] & W \\
W^* \arrow[u, "\phi^*"] \arrow[r, dashed, "(\cdot)^{f^*}"] & R^m \arrow[u, dashed, "((\phi)^f_e)^T(\cdot)"'] \\
R^n \arrow[u, dashed, "(\cdot)^{-1}_{e^*}"] &
\end{tikzcd}$$

Dual Reverses Composition

Proposition. Let U, V, W be R–modules, $\phi \in \mathrm{Hom}_R(U, V)$, $\psi \in \mathrm{Hom}_R(V, W)$. Then

$$(\psi \circ \phi)^* = \phi^* \circ \psi^* : W^* \to U^*.$$

Proof.

For $g \in W^*$ and $u \in U$,

$$\begin{aligned}((\psi \circ \phi)^* g)(u) =& g((\psi \circ \phi)(u)) = g(\psi(\phi(u))) = (\psi^* g)(\phi(u)) \\ =& (\phi^*(\psi^* g))(u).\end{aligned}$$

Since both sides define the same functional on every u, we have equality of maps $W^* \to U^*$. $\square$

Dual Reverses Composition

1. The sequence of maps on the vector spaces runs in the forward direction:

$$U \xrightarrow{\phi} V \xrightarrow{\psi} W$$

2. The sequence of the dual maps runs in the reverse direction:

$$W^* \xleftarrow{\psi^*} V^* \xleftarrow{\phi^*} U^*$$

The dual map ϕ^* is defined by $(\phi^* f)(u) = f(\phi(u))$, where $f \in V^*$ and $u \in U$. The composition rule for duals also reverses the order: $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

Canonical Evaluation Map $\iota : V \to V^{**}$

Definition. For any R–module V, define the *evaluation* (canonical) map

$$\iota : V \longrightarrow V^{**} = \mathrm{Hom}_R(V^*, R), \qquad \iota(v)(f) := f(v) \quad (f \in V^*).$$

$$\begin{tikzcd}
V \arrow[r, "\iota"] \arrow[dr, "f"'] & V^{**} \arrow[d, dashed, "(\cdot)(f)"] \\
 & R
\end{tikzcd}$$

Proposition

Finite-dimensional Case

If V is finite-dimensional, then ι is an isomorphism; we write $V \simeq V^{**}$ *canonically*.

Proof.

Fix a basis $e = (e_1, \ldots, e_n)$ with dual $e^* = (e^1, \ldots, e^n)$. Then $\iota(e_j)$ is the functional on V^* sending f to $f(e_j)$. In the dual basis, $\iota(e_j)$ corresponds to the coordinate row $(e^1(e_j), \ldots, e^n(e_j)) = (0, \ldots, 1, \ldots, 0)$, hence ι sends a basis to a basis and is therefore an isomorphism. Basis-independence: if we change basis by an invertible matrix E, the dual basis changes by E^{-1T}, and the resulting matrix of ι remains the identity; thus ι is canonical. $\square$

Naturality with Respect to Double Dual

Let V, W be finite-dimensional, and $\phi \in \mathrm{Hom}_R(V, W)$. The double dual map $\phi^{**} : V^{**} \to W^{**}$ is defined by

$$\phi^{**}(\Lambda) := \Lambda \circ \phi^*, \qquad (\Lambda \in V^{**}).$$

Then the square commutes:

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\iota_V"'] & W \arrow[d, "\iota_W"] \\ V^{**} \arrow[r, "\phi^{**}"'] & W^{**} \end{tikzcd}$$

Proof

Proof.

For $v \in V$ and $g \in W^*$,

$$\begin{aligned}(\phi^{**} \circ \iota_V)(v)(g) =& \iota_V(v)(\phi^* g) = (\phi^* g)(v) = g(\phi(v)) \\ =& \iota_W(\phi(v))(g) \\ =& (\iota_W \circ \phi)(v)(g).\end{aligned}$$

Thus $\phi^{**} \circ \iota_V = \iota_W \circ \phi$. $\square$

Bilinear Maps and Currying

Definition.

Let U, V, W be R–modules. A map $\Phi : U \times V \to W$ is *bilinear* if

$$\Phi(u_1 + u_2, v) = \Phi(u_1, v) + \Phi(u_2, v), \quad \Phi(\lambda u, v) = \lambda\Phi(u, v),$$

$$\Phi(u, v_1 + v_2) = \Phi(u, v_1) + \Phi(u, v_2), \quad \Phi(u, \lambda v) = \lambda\Phi(u, v).$$

Equivalently, the *curried* map $\widetilde{\Phi} : U \to \mathrm{Hom}_R(V, W)$,
$\widetilde{\Phi}(u)(v) = \Phi(u, v)$, is R–linear:

$$\Phi \in \mathcal{L}(U, V; W) \iff \widetilde{\Phi} \in \mathrm{Hom}_R\left(U, \mathrm{Hom}_R(V, W)\right).$$

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Examples

- Dot product (over a CRWU contained in $\mathbb{R}$): $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i : R^n \times R^n \to R$ is bilinear.
- Evaluation: eval $: V \times V^* \to R$, $\mathrm{eval}(v, f) = f(v)$ is bilinear since it is linear in each entry.

$$
\begin{tikzcd}
U \times V \arrow[r, "\Phi"] \arrow[d, dashed, "\text{curry}"'] & W \\
U \arrow[r, "\widetilde{\Phi}"'] & \mathrm{Hom}_R(V, W) \arrow[u, dashed, "\text{ev}"']
\end{tikzcd}
$$

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Thanks