Combinatorial Mesh Calculus (CMC): Lecture 10

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Diffeomorphisms

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Definition

Let $n \in \mathbb{N}$ and $U, V \subseteq \mathbb{R}^n$. We say that U and V are **diffeomorphic** if there exist smooth maps

$$f : U \to V, \quad g : V \to U$$

such that $g \circ f = \mathrm{id}_U$ and $f \circ g = \mathrm{id}_V$. We write $U \cong V$.

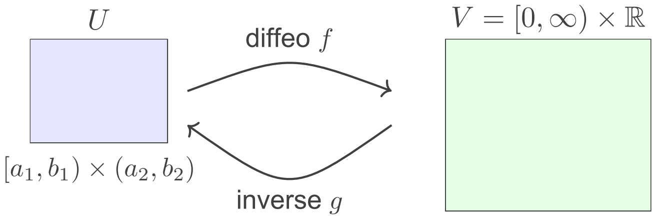
Examples

1. $U = (-\frac{\pi}{2}, \frac{\pi}{2})$, $V = \mathbb{R}$, $f = \tan$, $g = \arctan$; both smooth.
2. Any open n-brick $(a_1, b_1) \times \cdots \times (a_n, b_n)$ is diffeomorphic to $\mathbb{R}^n$.

Diffeomorphisms - Examples

Examples

3. $U = [0, \frac{\pi}{2})$, $V = \mathbb{R}_{\geq 0}$, tan : $U \to V$ is a diffeomorphism.
4. Generalization: $U = [a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$, $V = [0, \infty) \times \mathbb{R}^{n-1}$ are diffeomorphic by tan–like component maps.



Smooth Manifolds (with Boundary)

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Definition

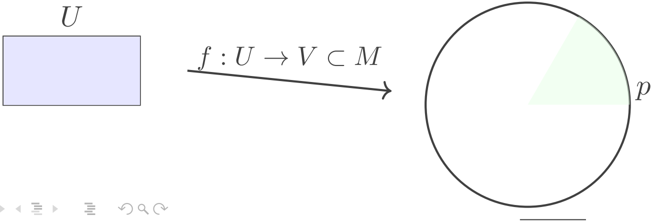
Let $m, n \in \mathbb{N}$ and $M \subseteq \mathbb{R}^{m+n}$. We say that M is a **smooth n–manifold (with boundary)** if for every $x \in M$ one of the following holds:

1. (*Interior point*) There exists an open neighbourhood V of x ($x \in V \subseteq M$) in M and an open set $U \subseteq \mathbb{R}^n$ together with a bijective immersion $f : U \to V$ such that Df has full rank n.
2. (*Boundary point*) There exists a neighbourhood V of x ($x \in V \subseteq M$) in M and $U \subseteq \mathbb{R}^n$ diffeomorphic to a half–space, and a bijective immersion $f : U \to V$.

Examples of Manifolds

Examples

1. Any open set $M \subseteq \mathbb{R}^n$ is a smooth n–manifold without boundary. For any $x \in M$ take $U = V = M$ $f = \mathrm{id}_M$.
2. $n = 2$, $M = \overline{B_1(0)} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ is a 2–manifold with boundary $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$.
3. For $p = (1, 0) \in S^1$, a chart can be built from a rectangle in parameter space mapping smoothly to a circular neighbourhood on M.



More Examples and Remarks

Examples

4 For any $n \in \mathbb{N}$,

$$B_1(0) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < 1\}$$

is a smooth n–manifold with boundary S^{n-1}.

5 The $(n-1)$–sphere

$$S^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$$

is a smooth $(n-1)$–manifold without boundary.

6 If M is a smooth n–manifold with boundary ∂M, then ∂M is a smooth $(n-1)$–manifold without boundary and $\partial(\partial M) = \varnothing$.

Local Chart on the Circle near $p = (1, 0)$

Note (efficient construction of a chart on S^1)

Let $M = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ and fix $p = (1, 0) \in S^1$. Choose angles $\alpha, \beta > 0$ with $\alpha + \beta < 2\pi$, and set

$$U := (-\alpha, \beta) \subset \mathbb{R}, \qquad V := \{(\cos\phi, \sin\phi) \mid \phi \in (-\alpha, \beta)\} \subset S^1.$$

Define the map

$$f : U \to V, \qquad f(\phi) = (\cos\phi, \sin\phi).$$

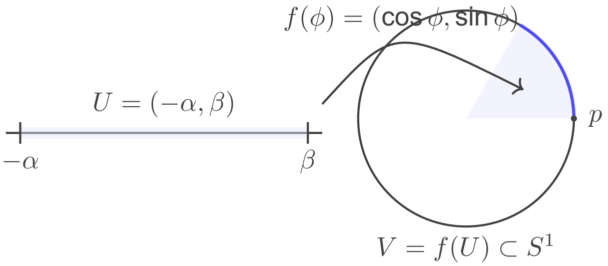
Then f is a bijective immersion: its Jacobian (column) is

$$Df(\phi) = \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for all } \phi \in U.$$

Local Chart on the Circle near $p = (1, 0)$

Construction of a chart on S^1)

Hence (U, f) is a smooth chart around $p = f(0)$. *Why V cannot be all of S^1:* the angle ϕ is 2π-periodic, so no single global parametrization $\phi \mapsto (\cos\phi, \sin\phi)$ is injective on all of S^1. One needs at least two overlapping charts (e.g. remove $(\pm 1, 0)$) to cover S^1.



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Local Charts on a Smooth Manifold

Definition

Let M be a smooth n–manifold. A **chart** (or local parametrization) on M is a smooth bijective immersion

$$f : U \to V, \quad U \subseteq_{\text{open}} M,$$

where

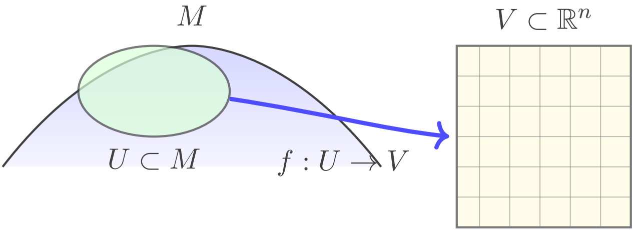
$$V \subseteq \begin{cases} \mathbb{R}^n, & \text{if } U \subseteq \text{Int}(M), \\ [0, \infty) \times \mathbb{R}^{n-1}, & \text{if } U \cap \partial M \neq \varnothing. \end{cases}$$

Both f and its inverse f^{-1} are smooth. The components of f define the *local coordinates* on U.

Local Charts on a Smooth Manifold

Intuition

Each chart provides a smooth "flattening" of a curved region of M into $\mathbb{R}^n$. Locally, the manifold looks like ordinary Euclidean space, even if globally it may curve or close on itself.



Atlases and Transition Maps

Definition

An **atlas** on a smooth manifold M is a family of charts

$$\{(U_i, f_i) \mid i \in I\}, \quad \bigcup_{i \in I} U_i = M,$$

such that all transition maps

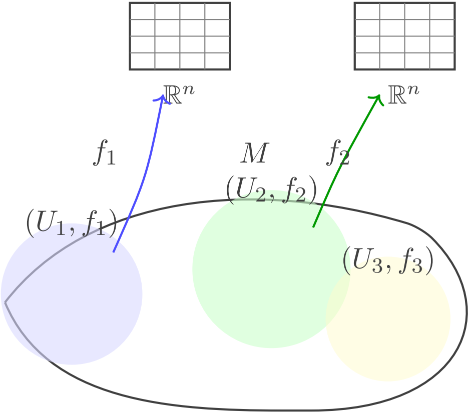
$$f_j \circ f_i^{-1} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$$

are smooth wherever the charts overlap. Different atlases that are compatible define the same smooth structure on M.

Atlases and Transition Maps

Geometric Idea

Each chart gives a local coordinate system, and their overlaps fit together smoothly. This collection allows calculus on M.



Example

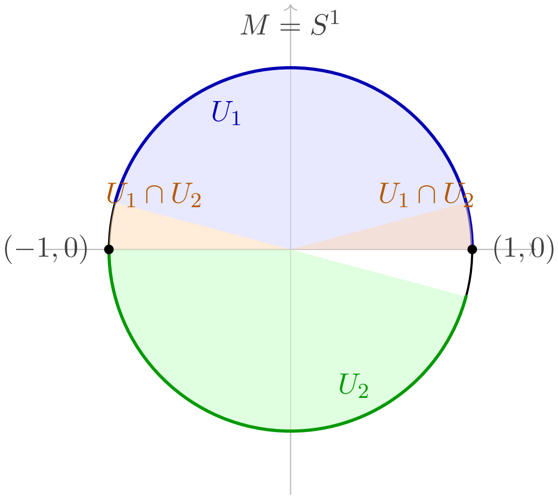
Construction

Let $M = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Since $f(\phi) = (\cos\phi, \sin\phi)$ is 2π–periodic, no single chart covers all of S^1. We define two overlapping charts:

$$U_1 = S^1 \setminus \{(1,0)\}, \qquad f_1(\phi) = (\cos\phi, \sin\phi), \quad \phi \in (0, 2\pi),$$
$$U_2 = S^1 \setminus \{(-1,0)\}, \quad f_2(\theta) = (\cos\theta, \sin\theta), \quad \theta \in (-\pi, \pi).$$

On the overlap $U_1 \cap U_2$, the transition map $f_2^{-1} \circ f_1$ is smooth, hence $\{(U_1, f_1), (U_2, f_2)\}$ forms an atlas.

Example



Smooth Maps between Manifolds

Definition

Let M, N be smooth manifolds with dim $M = m$ and dim $N = n$. A map $f : M \to N$ is **smooth** if for every chart (U, ϕ) on M and every chart (V, ψ) on N with $f(U) \subseteq V$, the composition

$$\psi \circ f \circ \phi^{-1} : \phi(U) \subseteq \mathbb{R}^m \to \mathbb{R}^n$$

is smooth in the classical sense. The space of all smooth maps is denoted $C^\infty(M, N)$. If $N = \mathbb{R}$, we write $\mathcal{F}(M) := C^\infty(M, \mathbb{R})$ — a commutative $\mathbb{R}$–algebra with unity.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\phi"'] & N \arrow[d, "\psi"] \\ \mathbb{R}^m \arrow[r, "\psi \circ f \circ \phi^{-1}"'] & \mathbb{R}^n \end{tikzcd}$$

Example: Inclusion $S^1 \hookrightarrow S^2$

Example

Let

$$M = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$
$$N = S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$

and define $f : M \to N$ by $f(x, y) = (x, y, 0)$. For charts:

$$\alpha : (E, F) \to U, \quad \alpha(\phi) = (\cos\phi, \sin\phi),$$
$$\beta : (A, B) \times (C, D) \to V, \quad \beta(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Then

$$\beta\left(\frac{\pi}{2}, \phi\right) = (\cos\phi, \sin\phi, 0) = f(\cos\phi, \sin\phi) = f(\alpha(\phi)),$$

showing the compatibility of local charts.

Example

$$
\begin{tikzcd}
 & U \subseteq S^1 \arrow[r, "f"] & V \subseteq S^2 & \\
(E,F) \arrow[ur, "\alpha"] \arrow[rrr, bend right, "x(\phi) = (\pi/2, \phi)"'] & & & (A,B) \times (C,D) \arrow[ul, "\beta"']
\end{tikzcd}
$$

Vector Fields on a Smooth Manifold

Definition

Let M be a smooth manifold. A **vector field** on M is an operator $X : \mathcal{F}(M) \to \mathcal{F}(M)$ satisfying:

1. $\forall f, g \in \mathcal{F}(M)$, $X(fg) = X(f)g + fX(g)$ (Leibniz rule)
2. X is $\mathbb{R}$–linear.

Examples

- $M = \mathbb{R}$, $X = \dfrac{d}{dx}$, then $X(fg) = f'g + fg'(= \frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x})$.
- $M = \mathbb{R}^n$, $X = \displaystyle\sum_{i=1}^{n} X^i(x)\frac{\partial}{\partial x_i}$.

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The Module of Vector Fields

Theorem

Let M be a smooth manifold and $\mathcal{X}(M)$ the set of vector fields on M. Define scalar multiplication by

$$(fX)(g) = f\,(Xg), \qquad f, g \in \mathcal{F}(M),\ X \in \mathcal{X}(M).$$

Then $(\mathcal{X}(M), +, \cdot)$ is an $\mathcal{F}(M)$–module.

Local Form

If (U, ϕ) is a chart with coordinates $(x_1, \dots, x_n)$, then

$$\mathcal{X}(U) \text{ is a free } \mathcal{F}(U)\text{–module with basis } \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}.$$

Every $X \in \mathcal{X}(U)$ can be expressed as

$$X = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}, \quad f_i \in \mathcal{F}(U).$$

Example: Euler Vector Field in $\mathbb{R}^2$

Example

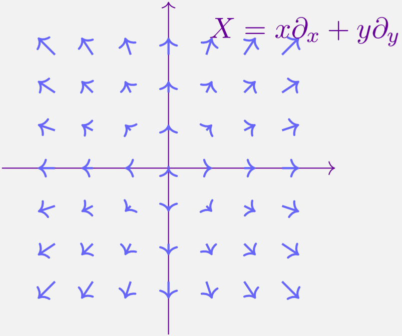
The **Euler vector field** on $\mathbb{R}^2$ is

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

For $f(x, y)$, we have

$$Xf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

Example: Euler Vector Field in $\mathbb{R}^2$



Differential Forms

Definition

Let M be a smooth manifold. The space of differential forms is

$$\Omega^\bullet(M) := \Lambda^\bullet((\mathcal{X}(M))^*).$$

If dim $M = n$, in local coordinates $(x_1, \ldots, x_n)$, the basis of $\mathcal{X}(U)$ is $\partial/\partial x_i$, and the dual basis of $(\mathcal{X}(U))^* = \Omega^1(U)$ is dx_i, satisfying

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}.$$

Differential Forms

Examples

- $\mathbb{R}^2$: $\Omega^0 M = \mathcal{F}(M)$,
  - $\Omega^1 M = \{f_x dx + f_y dy\}$,
  - $\Omega^2 M = \{g\, dx \wedge dy\}$.
- $\mathbb{R}^3$:
  - $\Omega^0 M = \mathcal{F}(M)$,
  - $\Omega^1 M = \{f_1 dx + f_2 dy + f_3 dz\}$,
  - $\Omega^2 M = \{g_1 dy \wedge dz + g_2 dz \wedge dx + g_3 dx \wedge dy\}$,
  - $\Omega^3 M = \{h\, dx \wedge dy \wedge dz\}$.

Exterior Derivative

Definition

For a smooth manifold M, the exterior derivative is the graded operator

$$d : \Omega^\bullet(M) \to \Omega^\bullet(M), \quad d_p : \Omega^p(M) \to \Omega^{p+1}(M),$$

defined by:

1. For $f \in \Omega^0(M) = \mathcal{F}(M)$,
$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$
2. For $\omega, \eta \in \Omega^\bullet(M)$,
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\text{deg}\,\omega} \omega \wedge d\eta.$$

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Exterior Derivative

Definition

3 $d_{p+1} \circ d_p = 0$ (cochain complex property).

$$\Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \xrightarrow{d_2} \Omega^3(M)$$

Example: Gradient and Curl in $\mathbb{R}^2$

Example

Let $M = \mathbb{R}^2$ and $f \in \Omega^0(M)$, then

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = (\nabla f)^\flat.$$

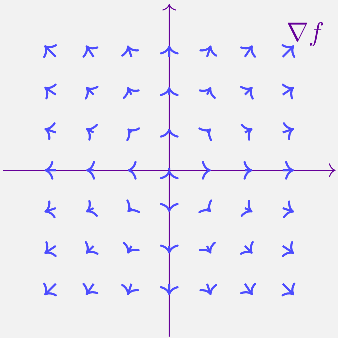
For $\omega = P\,dx + Q\,dy \in \Omega^1(M)$,

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy,$$

which corresponds to the curl of (P, Q).

Example: Gradient and Curl in $\mathbb{R}^2$

Example



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Summary of Lecture

- Introduced diffeomorphisms and smooth manifolds.
- Described examples: disks, spheres, and general n–bricks as manifolds or manifolds with corners.
- Defined charts, local coordinates, and atlases with geometric illustrations.
- Demonstrated circle parametrization and Jacobian regularity for immersion.
- Defined smooth maps between manifolds using charts and commutative diagrams.
- Introduced vector fields as derivations and showed $\mathcal{X}(M)$ is an $\mathcal{F}(M)$–module. Illustrated the Euler vector field on $\mathbb{R}^2$ geometrically.
- Defined differential forms, and exterior derivative. Showed $d_{p+1} \circ d_p = 0$, relating gradients and curls in $\mathbb{R}^2$ to Green's theorem.

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Thanks