Combinatorial Mesh Calculus (CMC): Lecture 11

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Coordinate Patch in Polar Form

Example - Setting

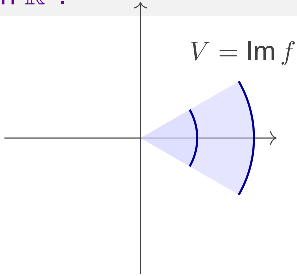
Let $U = (\frac{1}{2}, 1] \times (-\frac{\pi}{6}, \frac{\pi}{6})$ and define

$$f : U \to \mathbb{R}^2, \qquad f(r, \phi) = (r \cos \phi, r \sin \phi).$$

Then the image

$$V = \operatorname{Im} f = \{(r \cos \phi, r \sin \phi) \mid r \in (\tfrac{1}{2}, 1], \phi \in (-\tfrac{\pi}{6}, \tfrac{\pi}{6})\}$$

is an annular sector in $\mathbb{R}^2$.



Exterior Derivative in $\mathbb{R}^3$

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Definition

For a smooth manifold $M = \mathbb{R}^3$, the exterior derivative (ED)

$$d_p : \Omega^p M \to \Omega^{p+1} M$$

satisfies $d_{p+1} \circ d_p = 0$ and the graded Leibniz rule.

Concrete Computations

Let $f : \mathbb{R}^3 \to \mathbb{R}$.

$$d_0 f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

For a 1-form $\omega = f_x dx + f_y dy + f_z dz$,

Example

Concrete Computations

$$
\begin{aligned}
d_1\omega &= d(f_x) \wedge dx + d(f_y) \wedge dy + d(f_z) \wedge dz \\
&= (f_{zy} - f_{yz})\, dy \wedge dz + (f_{xz} - f_{zx})\, dz \wedge dx + (f_{yx} - f_{xy})\, dx \wedge dy.
\end{aligned}
$$

Thus $d_1\omega$ corresponds to curl(f_x, f_y, f_z).

For a 2-form

If $\eta = f_x\, dy \wedge dz + f_y\, dz \wedge dx + f_z\, dx \wedge dy$, then

$$d_2\eta = (f_{x,x} + f_{y,y} + f_{z,z})\, dx \wedge dy \wedge dz,$$

which represents the divergence $\nabla \cdot (f_x, f_y, f_z)$.

Differential Operators as ED

In $\mathbb{R}^2$

$$d_0 f = \nabla f = (f_x, f_y),$$

$$d_1(f_x dx + f_y dy) = (f_{yx} - f_{xy}) dx \wedge dy \Rightarrow \text{scalar curl},$$

$$d_1 \circ d_0 = 0 \;\Rightarrow\; \text{curl}(\nabla f) = 0.$$

$$\begin{array}{c} \mathcal{F}(\mathbb{R}^2) = \Omega^0 \\ \big\downarrow d_0 = \nabla \\ \mathcal{X}(\mathbb{R}^2) = \Omega^1 \\ \big\downarrow d_1 = \text{curl} \\ \Omega^2(\mathbb{R}^2) \end{array}$$

Differential Operators as ED

In $\mathbb{R}^3$

$$d_0 f = \nabla f,$$

$$d_1(f_x dx + f_y dy + f_z dz) = \text{curl}(f_x, f_y, f_z),$$

$$d_2(f_x dy \wedge dz + \dots) = \text{div}(f_x, f_y, f_z),$$

$$d_2 \circ d_1 = 0 \;\Rightarrow\; \nabla \cdot (\nabla \times A) = 0.$$

$$\begin{array}{c}
\mathcal{F}(\mathbb{R}^3) = \Omega^0 \\
\big\downarrow d_0 = \nabla \\
\mathcal{X}(\mathbb{R}^3) = \Omega^1 \\
\big\downarrow d_1 = \text{curl} \\
\Omega^2(\mathbb{R}^3) \\
\big\downarrow d_2 = \text{div} \\
\Omega^3(\mathbb{R}^3)
\end{array}$$

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Pullback of Differential Forms

Definition

Let M, N be smooth manifolds and $f \in C^\infty(M, N)$. For a 1-form $\omega \in \Omega^1 N$ with local expression

$$\omega = h_1\, dy_1 + \cdots + h_n\, dy_n, \quad h_i \in \mathcal{F}(N),$$

the pullback $f^*\omega \in \Omega^1 M$ is defined by

$$f^*\omega = (h_1 \circ f)\, d(y_1 \circ f) + \cdots + (h_n \circ f)\, d(y_n \circ f).$$

It extends naturally to exterior powers: $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

Pullback of Differential Forms

Example

Let $M = N = \mathbb{R}^2$, $f(r,\phi) = (r\cos\phi, r\sin\phi)$, and $\omega = -y\,dx + x\,dy$. Then

$$\begin{aligned} f^*\omega =& -(r\sin\phi)\,d(r\cos\phi) + (r\cos\phi)\,d(r\sin\phi) \\ =& -(r\sin\phi)(\cos\phi\,dr - r\sin\phi\,d\phi) \\ & + (r\cos\phi)(\sin\phi\,dr + r\cos\phi\,d\phi) \\ =& r^2\,d\phi. \end{aligned}$$

Compatibility of Pullback and ED

Theorem (Compatibility of Pullback and Exterior Derivative)

Let M, N be smooth manifolds and $f \in C^\infty(M, N)$ a smooth map. For each degree $p \in \mathbb{N}$, the pullback operator

$$f_p^* : \Omega^p(N) \to \Omega^p(M)$$

commutes with the exterior derivative, i.e.

$$d_p^M \circ f_p^* = f_{p+1}^* \circ d_p^N .$$

In other words, taking the pullback of a form and then differentiating gives the same result as differentiating first and then taking the pullback:

$$f^*(d\omega) = d(f^*\omega), \quad \forall\, \omega \in \Omega^p(N).$$

This shows that the exterior derivative is a *natural operator* with respect to smooth maps between manifolds.

Example

$$\begin{tikzcd}
\Omega^p(N) \arrow[r, "d_p^N"] \arrow[d, "f_p^*"'] & \Omega^{p+1}(N) \arrow[d, "f_{p+1}^*"] \\
\Omega^p(M) \arrow[r, "d_p^M"'] & \Omega^{p+1}(M)
\end{tikzcd}$$

Example

Let $M = N = \mathbb{R}^2$, $f(r,\phi) = (r\cos\phi, r\sin\phi)$, $\omega = dx \wedge dy$:

$$f^* dx = \cos\phi\, dr - r\sin\phi\, d\phi,$$
$$f^* dy = \sin\phi\, dr + r\cos\phi\, d\phi,$$
$$f^*(dx \wedge dy) = (f^* dx) \wedge (f^* dy) = r\, dr \wedge d\phi.$$

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Trace of Differential Forms on Submanifolds

Definition

Let M be a smooth manifold and $N \subseteq M$ a submanifold with inclusion map

$$\iota_N : N \hookrightarrow M, \quad \iota_N(x) = x.$$

The **trace** (restriction) of forms is the pullback

$$\mathrm{tr}_N = \iota_N^* : \Omega^\bullet M \to \Omega^\bullet N.$$

If $N \subseteq \partial M$, this represents the restriction of forms to the boundary.

Example

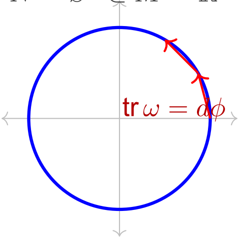
Example

Let $M = \mathbb{R}^2$, $N = S^1 \subseteq M$, and $\omega = x\,dy - y\,dx$. The inclusion $\iota_{S^1} : S^1 \to \mathbb{R}^2$ gives

$$\begin{aligned} \operatorname{tr}_{S^1}\omega =& (x\,dy - y\,dx)\big|_{S^1} \\ =& (r\cos\phi)(r\cos\phi\,d\phi) - (r\sin\phi)(-r\sin\phi\,d\phi) = r^2 d\phi, \end{aligned}$$

and for $r = 1$, $\operatorname{tr}_{S^1}\omega = d\phi$.

$N = S^1 \subset M = \mathbb{R}^2$



Orientation of Smooth Manifolds

Definition

Let M be a smooth manifold of dimension D. Since the space of top–degree forms $\Omega^D M$ is locally 1–dimensional, every element $\omega \in \Omega^D M$ can be written locally as

$$\omega = f\,dx_1 \wedge dx_2 \wedge \cdots \wedge dx_D, \qquad f \in \mathcal{F}(M).$$

If $f(x) \neq 0$ for all $x \in M$, then ω is said to be a **nonvanishing D–form**, and M is called **orientable**. A choice of such an ω (and identifying all positive scalar multiples of ω as equivalent) defines an **orientation** of M.

If M is connected and orientable, it admits exactly two possible orientations, represented by ω and $-\omega$. If M is disconnected, each connected component may be oriented independently.

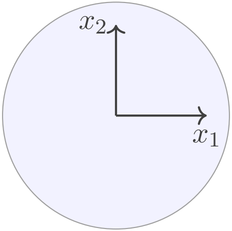
Orientation of Smooth Manifolds

Intuitively, an orientation distinguishes between the two possible "directions of measurement" on M: one associated with ω (positive orientation) and the other with $-\omega$ (negative orientation).

Example

$\mathbb{R}^D$ is orientable with orientation

$$\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_D, \quad \text{or its opposite} \quad -dx_1 \wedge \cdots \wedge dx_D.$$



$\omega = dx_1 \wedge dx_2$ gives positive orientation

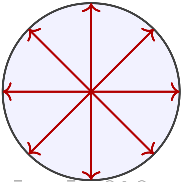
Outward-Pointing Vector Fields

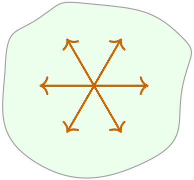
Definition

Let $M \subseteq \mathbb{R}^D$ be a D–dimensional manifold with boundary. A vector field $X \in \mathcal{X}(M)$ is called **outward-pointing** on ∂M if near every boundary point it locally points outside M.

Examples

- $M = \overline{B}_1(0,0) \subseteq \mathbb{R}^2, \quad X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y};$
- $M = \overline{B}_1(0,0,0) \subseteq \mathbb{R}^3, \quad X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$





The Induced Orientation on Boundaries

Theorem (Induced Orientation)

Let M be an orientable D–manifold with boundary $S = \partial M$ and orientation form $\omega \in \Omega^D M$. If $X \in \mathcal{X}(M)$ is outward-pointing, then

$$\omega_S = \mathrm{tr}_{\partial M}(\iota_X \omega) \in \Omega^{D-1}(S)$$

defines an orientation form on S, called the **induced orientation**.

Example

Example 1: Half–Plane

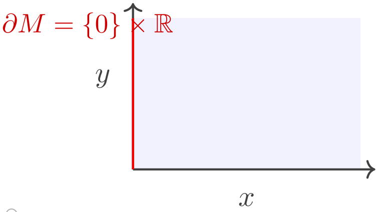
Let $M = [0, \infty) \times \mathbb{R}$, $S = \{0\} \times \mathbb{R}$,

$$X = -\frac{\partial}{\partial x}, \quad \omega = dx \wedge dy.$$

Then

$$\iota_X (dx \wedge dy) = \iota_{-\partial_x} (dx \wedge dy) = -\,\iota_{\partial_x} (dx \wedge dy) = -\,dy.$$

Thus $\omega_S = -dy$: the boundary inherits the leftward orientation.



Induced Orientation in 2D and 3D Balls

Example 2: Disk in $\mathbb{R}^2$

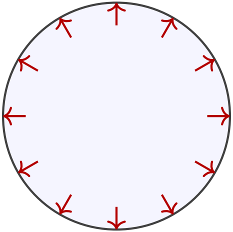
$$M = B_1(0,0),\ \partial M = S^1,\ \ X = x\partial_x + y\partial_y, \qquad \omega_M = dx \wedge dy.$$

Then $\iota_X \omega_M = \iota_{x\partial_x + y\partial_y}(dx \wedge dy) = x\,dy - y\,dx.$

In polar coordinates $x = r\cos\phi,\ y = r\sin\phi$, so

$$x\,dy - y\,dx = r^2\,d\phi \quad \Rightarrow \quad \mathrm{tr}_{S^1}(\iota_X \omega_M) = d\phi.$$

Hence, S^1 inherits counterclockwise orientation.



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3D Ball: Induced Orientation on the Sphere

Example 3: Ball in $\mathbb{R}^3$

$M = B_1(0,0,0)$, $\partial M = S^2$,

$$X = x\partial_x + y\partial_y + z\partial_z, \qquad \omega_M = dx \wedge dy \wedge dz.$$

Then

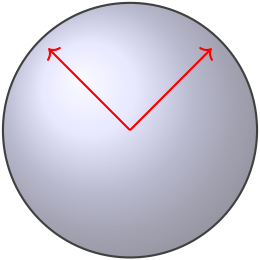
$$\iota_X \omega_M = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy.$$

In spherical coordinates
$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$, we have

$$\mathrm{tr}_{S^2}(\iota_X \omega_M) = r^2 \sin\theta\, d\theta \wedge d\phi.$$

This 2–form defines the standard orientation of S^2.

Visualization



$$\omega_S = \sin\theta \, d\theta \wedge d\phi$$

Spherical Coordinates

Spherical chart on S^2 (radius 1)

Standard spherical coordinates

$x = \sin\theta\cos\phi, \quad y = \sin\theta\sin\phi, \quad z = \cos\theta, \theta \in (0,\pi), \ \phi \in (0, 2\pi)$

yield the induced orientation 2–form

$$\omega_{S^2} = \sin\theta \, d\theta \wedge d\phi.$$

Here $d\theta \wedge d\phi$ is ordered so that $(\partial_\theta, \partial_\phi)$ agrees with the outward normal orientation. $\sin\theta$ is the Jacobian density (area scale factor) of the chart; it vanishes only at the poles $\theta = 0, \pi$, where this chart is *singular* (coordinate singularity, not a geometric one).

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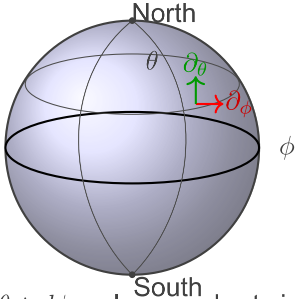
Visualization



$\omega_{S^2} = \sin\theta\, d\theta \wedge d\phi$; poles are chart singularities.

Curvature Intuition on S^2: Meridians vs Parallels

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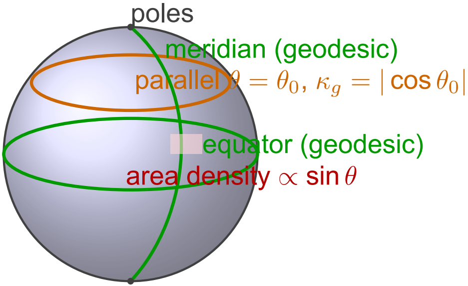
Geodesics and geodesic curvature (unit sphere)

- **Meridians** ($\phi =$ const) and **equator** ($\theta = \pi/2$) are *great circles* $\Rightarrow$ geodesics with geodesic curvature $\kappa_g = 0$.
- **Parallels** ($\theta =$ const $\neq \pi/2$) are not geodesics; their geodesic curvature is $\kappa_g = |\cos\theta|$ (nonzero away from the equator).
- The area element (orientation form) is $\omega_{S^2} = \sin\theta \, d\theta \wedge d\phi$: bands near the equator ($\theta \approx \pi/2$) have greater area density; near the poles ($\theta \to 0, \pi$) the density vanishes.

Visualization



Summary

The θ, ϕ chart encodes orientation and area via $\sin\theta$, is singular at the poles, and separates geodesic directions (great circles) from curved parallels (nonzero κ_g).

Compact Manifolds and Integration

Definition

A manifold $M \subseteq \mathbb{R}^D$ is **compact** if it is closed and bounded, i.e. it lies entirely inside some finite ball $B_R(0)$.

Examples

- Closed D–bricks and their boundaries,
- Closed D–balls $\overline{B}_1(0)$ and spheres S^{D-1}.

Integration

Smooth Oriented D–Manifolds

For compact, oriented M, integration is a linear map

$$\int_M : \Omega^D M \to \mathbb{R}$$

satisfying:

1. **Additivity:** If $M = M_1 \cup M_2$ with same orientation,
$$\int_M \omega = \int_{M_1} \mathrm{tr}_{M_1}\, \omega + \int_{M_2} \mathrm{tr}_{M_2}\, \omega.$$
2. **Change of Variables:** If $\phi : M \to N$ is an orientation–preserving diffeomorphism,
$$\int_N \omega = \int_M \phi^* \omega.$$

Integration

Smooth Oriented D–Manifolds

3 **Stokes–Cartan Theorem:**

$$\int_M d\omega = \int_{\partial M} \mathrm{tr}_{\partial M}\, \omega.$$

4 **Zero–Dimensional Case:** If $M = \{x\}$ with orientation $\epsilon = \pm 1$, $\displaystyle\int_{\{x\}} f = \epsilon\, f(x)$.

Example 1: $(D = 1)$

Statement: Newton–Leibniz Theorem

Let $M = [a, b] \subset \mathbb{R}$, with boundary $\partial M = \{a, b\}$ oriented as $a \mapsto b$.
For $f \in C^\infty(M)$,

$$\omega = df = f'(x)\, dx.$$

Then, by Stokes–Cartan:

$$\int_{[a,b]} df = \int_{\partial[a,b]} \mathrm{tr}\, f = f(b) - f(a).$$

$$\int_a^b f'(x)\, dx = f(b) - f(a)$$

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Example 2: ($D = 2$, Curve in $\mathbb{R}^2$)

Statement: Gradient Theorem

Let $\gamma : [a, b] \to \mathbb{R}^2$ be a smooth oriented 1–manifold with boundary $\partial\gamma = \{\gamma(a), \gamma(b)\}$, and $f \in C^\infty(\mathbb{R}^2)$. The 1–form

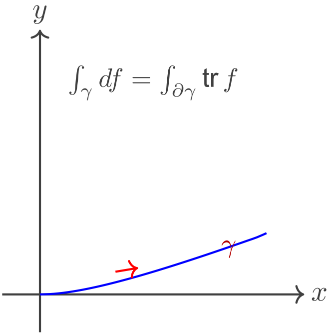
$$\omega = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Then

$$\int_\gamma df = \int_{\partial\gamma} \mathrm{tr}_{\partial\gamma} f = f(\gamma(b)) - f(\gamma(a)).$$

Thus, the **gradient theorem** is the 2–dimensional instance of Stokes–Cartan.

Visualization



Example 3: $(D = 2)$

Statement: Green's Theorem

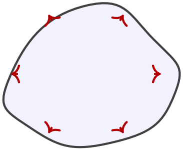
Let $M \subset \mathbb{R}^2$ be a compact oriented 2–manifold with boundary ∂M (positively oriented). For a 1–form $\omega = P\,dx + Q\,dy$,

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

Then by Stokes–Cartan,

$$\int_M d\omega = \int_{\partial M} \mathrm{tr}_{\partial M}\,\omega.$$

This is **Green's theorem** in exterior–form form: the flux of $d\omega$ across M equals the trace of ω on ∂M.



Example 4: $D = 3$

Statement (Stokes–Cartan for a 1–manifold in $\mathbb{R}^3$)

Let $\gamma : [a, b] \to \mathbb{R}^3$ be a smooth oriented curve (a compact 1–manifold $M = \gamma([a, b])$ with $\partial M = \{\gamma(a), \gamma(b)\}$ and orientation from a to b). For $f \in \mathcal{F}(\mathbb{R}^3) = C^\infty(\mathbb{R}^3)$, the 1–form

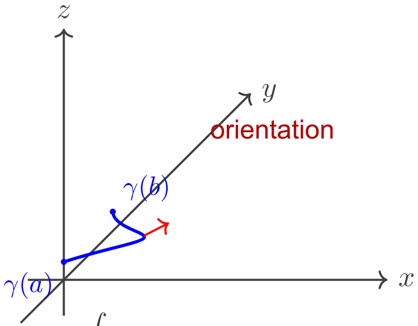
$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz.$$

Then, by Stokes–Cartan on the 1–manifold M,

$$\boxed{\int_\gamma df = \int_{\partial\gamma} \mathrm{tr}_{\partial\gamma} f = f(\gamma(b)) - f(\gamma(a))}$$

(i.e. the *line integral of the exact form* df along the space curve equals the trace of f on the boundary points).

Visualization



$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a))$$

Example 5: ($D = 3$)

Statement: Kelvin–Stokes Theorem

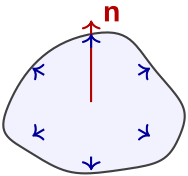
Let $M \subset \mathbb{R}^3$ be a smooth oriented 2–manifold with boundary ∂M.
For a 1–form $\omega = A\,dx + B\,dy + C\,dz$,

$$d\omega = \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy\wedge dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz\wedge dx + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx\wedge dy.$$

Then by Stokes–Cartan:

$$\int_M d\omega = \int_{\partial M} \mathrm{tr}_{\partial M}\, \omega.$$

This is the **Kelvin–Stokes theorem** in exterior form notation.



Example 6: $(D = 3)$

Statement: Gauss Divergence Theorem

Let $M \subset \mathbb{R}^3$ be a compact oriented 3–manifold with boundary ∂M. For a 2–form

$$\omega = P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy,$$

we have

$$d\omega = (\partial_x P + \partial_y Q + \partial_z R)\, dx \wedge dy \wedge dz.$$

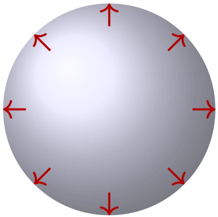
Then, by Stokes–Cartan:

$$\int_M d\omega = \int_{\partial M} \mathrm{tr}_{\partial M}\, \omega.$$

This is the **Gauss (Divergence) theorem** in the language of differential forms.

Visualization

outward orientation



$$\int_M d\omega = \int_{\partial M} \mathrm{tr}_{\partial M}\, \omega$$

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Summary: Stokes–Cartan in all Dim.

Unified Framework

For any compact oriented smooth D–manifold M with boundary ∂M:

$$\boxed{\int_M d\omega = \int_{\partial M} \mathrm{tr}_{\partial M}\, \omega.}$$

All classical theorems follow as special cases:

Dim.	**Differential Form**	**Result**
1	df	Newton–Leibniz theorem
2	df on γ	Gradient theorem
2	$d(P\,dx + Q\,dy)$	Green's theorem
3	$d(A\,dx + B\,dy + C\,dz)$	Kelvin–Stokes theorem
3	$d(P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy)$	Gauss divergence theorem

Part I: Lecture Summary

Summary

- The exterior derivative $d_p : \Omega^p M \to \Omega^{p+1} M$ generalizes grad, curl, and divergence.
- In $\mathbb{R}^2$: d_0 = grad, d_1 = scalar curl; in $\mathbb{R}^3$: d_0 = grad, d_1 = curl, d_2 = div.
- Pullback f^* transfers forms from N to M compatibly: $d \circ f^* = f^* \circ d$.
- Trace $\mathrm{tr}_N = \iota_N^*$ restricts forms to submanifolds or boundaries.
- These constructions make differential forms coordinate-independent tools for calculus on manifolds.

Part II: Lecture Summary

Summary

- Orientation is given by a nonvanishing top–degree form $\omega \in \Omega^D M$.
- Outward vector fields define induced orientation on ∂M via $\omega_S = \mathrm{tr}_{\partial M}(\iota_X \omega)$.
- Integration of differential forms generalizes classical calculus results to manifolds.
- Stokes–Cartan theorem unifies Newton–Leibniz, Green, Kelvin–Stokes, and Gauss divergence theorems.
- Compact oriented manifolds admit consistent integration respecting additivity and diffeomorphism invariance.

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