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Combinatorial Mesh Calculus (CMC): Lecture 13

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Integration on Manifolds

Definition

Let (M, g) be a compact oriented D–dimensional Riemannian manifold.

1. The **measure** of M (length, area, or volume for $D = 1, 2, 3$) is
$$\mu(M) = \int_M \mathsf{vol}_{(M,g)} > 0.$$
2. The **Riemann integral** of a smooth function $f \in \mathcal{F}(M)$ is
$$I(f) = \int_M f\, \mathsf{vol}_{(M,g)}.$$

Definition

For $D = 1$, $M = [a, b]$, this reduces to the classical integral

$$I(f) = \int_a^b f(x)\, dx.$$

3 The **inner product** of two p–forms $\omega, \eta \in \Omega^p(M)$ with $p \in \{0, 1, \ldots, D\}$, is

$$\begin{aligned}\langle \omega, \eta \rangle_p :=& I(g_p(\omega, \eta)).\\ =& \int_M g_p^*(\omega, \eta)\,\mathrm{vol}_{(M,g)}.\end{aligned}$$

Example: Area on the Sphere S_r^2

Setup

Let $D = 2$, $M = S_r^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = r^2\}$ with the induced orientation and metric from $\mathbb{R}^3$. In spherical coordinates:

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r\cos\theta,$$
$$\theta \in (0, \pi), \qquad \phi \in (0, 2\pi).$$

The induced metric is $g = r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$, and the volume form is

$$\mathrm{vol}_{(M,g)} = r^2 \sin\theta \, d\theta \wedge d\phi.$$

Example

Computation of Area of a Spherical Quadrilateral

For $\theta \in [\theta_1, \theta_2]$ and $\phi \in [\phi_1, \phi_2]$,

$$\begin{aligned} A &= \int_{[\theta_1,\theta_2]\times[\phi_1,\phi_2]} r^2 \sin\theta \, d\theta \wedge d\phi \\ &= r^2 \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \sin\theta \, d\theta \, d\phi \\ &= r^2 (\phi_2 - \phi_1)(\cos\theta_1 - \cos\theta_2). \end{aligned}$$

This agrees with the Frobenius theorem, ensuring the integration of a 2–form depends only on the orientation of M.

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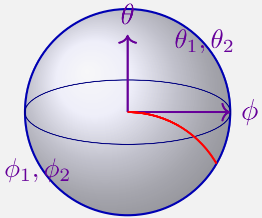
Visualization



Codifferential Operator Manifolds

Definition

Let (M, g) be a compact oriented D–dimensional Riemannian manifold, and $p \in \{1, 2, \ldots, D\}$. Denote by $\overset{\circ}{\Omega}{}^{p}(M) = \{\omega \in \Omega^p(M) \mid \mathrm{tr}_{\partial M}\,\omega = 0\}$ the space of p–forms vanishing on the boundary.
The **codifferential**

$$d_p^* : \overset{\circ}{\Omega}{}^{p}(M) \longrightarrow \overset{\circ}{\Omega}{}^{p-1}(M)$$

is the adjoint of d_{p-1} restricted to $\overset{\circ}{\Omega}{}^{p-1}(M)$:

$$\langle d_p^*\omega, \eta\rangle_{p-1} = \langle \omega, d_{p-1}\eta\rangle_p, \qquad \forall \omega \in \overset{\circ}{\Omega}{}^{p}(M),\ \eta \in \overset{\circ}{\Omega}{}^{p-1}(M).$$

Expression via Hodge Star

Equivalent Expression

For any $p \in \{1, \ldots, D\}$,

$$d^*_{D-p} \circ \star_p = (-1)^{p+1} \star_{p+1} \circ d_p,$$

or equivalently: $$d^*_{D-p} = (-1)^{p+1} \star_{p+1} \circ d_p \circ \star_p^{-1}$$
$$= (-1)^{p+1+p(D-p)} \star_{p+1} \circ d_p \circ \star_{D-p}.$$

$$\begin{tikzcd} \Omega^p(M) \arrow[r, "d_p"] \arrow[d, "\star_p"'] & \Omega^{p+1}(M) \arrow[d, "\star_{p+1}"] \\ \Omega^{D-p}(M) \arrow[r, "(-1)^{p+1} d^*_{D-p}"'] & \Omega^{D-p-1}(M) \end{tikzcd}$$

Codifferential: General Identity

Setup and Symbols

(M, g): oriented D-dimensional Riemannian manifold.

$\Omega^p(M)$: p-forms on M,

$d_p : \Omega^p(M) \to \Omega^{p+1}(M)$.

$\star_p : \Omega^p(M) \to \Omega^{D-p}(M)$ (Hodge star),

$\star_p^{-1} = (-1)^{p(D-p)} \star_{D-p}$.

Inner product: $g_p^*(\omega, \eta)\,\mathrm{vol}_g = \omega \wedge \star_p \eta$,

$\langle \omega, \eta \rangle_p = \int_M g_p^*(\omega, \eta)\,\mathrm{vol}_g$.

Adjoint of d

Codifferential

For $p \in \{0, 1, \ldots, D-1\}$, the adjoint

$$d^*_{D-p} = (-1)^{p+1+p(D-p)} \, \star_{p+1} \circ d_p \circ \star_{D-p},$$
$$d^*_{D-p} : \Omega^{D-p}(M) \longrightarrow \Omega^{D-p-1}(M).$$

$f \in \Omega^0(M) = \mathcal{F}(M)$ (smooth scalar function).

$\omega \in \Omega^1(M)$ (one-form; via g corresponds to a vector field by $\omega = X^\flat$).

$\eta \in \Omega^2(M)$ (two-form; in $D = 3$, $\eta = \star_1 Y^\flat$ encodes an oriented flux field).

Specialization to $D = 3$: Explicit d and d^*

Exterior derivatives in $D = 3$

$$
\begin{aligned}
d_0 &: \Omega^0(M) \to \Omega^1(M), & d_0(f) &= df, \\
d_1 &: \Omega^1(M) \to \Omega^2(M), & d_1(\omega) &= d\omega, \\
d_2 &: \Omega^2(M) \to \Omega^3(M), & d_2(\eta) &= d\eta, \\
d_3 &: \Omega^3(M) \to \Omega^4(M) = 0, & d_3 &= 0.
\end{aligned}
$$

Adjoints from the general identity

Using $d^*_{D-p} = (-1)^{p+1+p(D-p)} \star_{p+1}\ d_p \star_{D-p}$ with $D = 3$:

$$
\text{for } p = 2: \quad d^*_1 = (-1)^{2+1+2(1)} \star_3\ d_2 \star_1 = (-1)^5 \star_3\ d_2 \star_1 = -\star_3\ d_2 \star_1,
$$

Specialization to $D = 3$: Explicit d and d^*

Adjoints from the general identity

$d_1^* : \Omega^1(M) \to \Omega^0(M)$ ("negative divergence" under the standard identifications).

for $p = 1$:
$$\begin{aligned} d_2^* &= (-1)^{1+1+1(2)} \star_2 \, d_1 \star_2 \\ &= (-1)^4 \star_2 \, d_1 \star_2 \\ &= \star_2 \, d_1 \star_2 \end{aligned}$$

$d_2^* : \Omega^2(M) \to \Omega^1(M)$ ("curl" under the standard identifications).

for $p = 0$:
$$\begin{aligned} d_3^* &= (-1)^{0+1+0} \star_1 \, d_0 \star_3 \\ &= - \star_1 \, d_0 \star_3 , \end{aligned}$$

$d_3^* : \Omega^3(M) \to \Omega^2(M)$ (co-gradient of a density 3-form).

Interpretation in $\mathbb{R}^3$ and Schematic

Vector–calculus dictionary (with this sign convention)

$$
\begin{aligned}
d_0 &\rightsquigarrow \text{gradient } (\nabla f),\\
d_1 &\rightsquigarrow \text{curl } (\nabla \times A),\\
d_2 &\rightsquigarrow \text{divergence } (\nabla \cdot F),\\
d_1^* &= -\star_3 d_2 \star_1 \rightsquigarrow -\operatorname{div}(A),\\
d_2^* &= \star_2 d_1 \star_2 \rightsquigarrow \nabla \times F,\\
d_3^* &= -\star_1 d_0 \star_3 .
\end{aligned}
$$

$$
\begin{tikzcd}[column sep=huge]
\Omega^0 \arrow[r, "\overset{\text{grad}}{d_0}"] & \Omega^1 \arrow[l, bend left, "\substack{d_1^* = -\star_3 d_2 \star_1 \\ -\text{div}}"] \arrow[r, "\overset{\text{curl}}{d_1}"] & \Omega^2 \arrow[l, bend left, "\substack{d_2^* = \star_2 d_1 \star_2 \\ \text{curl}}"] \arrow[r, "\overset{\text{div}}{d_2}"] & \Omega^3 \arrow[l, bend left, "\substack{d_3^* = -\star_1 d_0 \star_3 \\ \text{co-grad}}"]
\end{tikzcd}
$$

Domains/Codomains

$D = 3$

$$f \in \Omega^0 \xrightarrow{d_0} df \in \Omega^1,$$

$$\omega \in \Omega^1 \xrightarrow{d_1} d\omega \in \Omega^2,$$

$$\eta \in \Omega^2 \xrightarrow{d_2} d\eta \in \Omega^3,$$

$$\alpha \in \Omega^1 \xrightarrow{d_1^*} d_1^*\alpha \in \Omega^0,$$

$$\beta \in \Omega^2 \xrightarrow{d_2^*} d_2^*\beta \in \Omega^1,$$

$$\gamma \in \Omega^3 \xrightarrow{d_3^*} d_3^*\gamma \in \Omega^2.$$

Setup and Field Quantities

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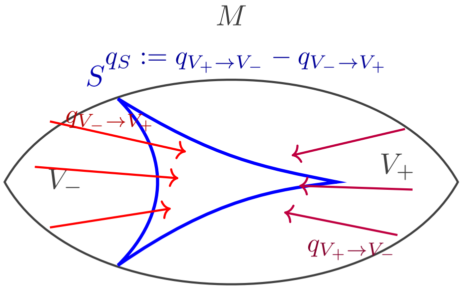
Geometric–Physical Setting

Let $D \in \mathbb{N}$. Let (M, g) be a compact, oriented D–dimensional Riemannian manifold with nonempty boundary ∂M (spatial domain). Let $t_0 \in \mathbb{R}$ and $I = [t_0, \infty)$ (time interval). We study the transport of an extensive quantity (mass, charge, energy, …) in M over I.

Differential–Form Fields (time–dependent)

Amount (density D–form): $Q \in C^\infty\big(I, \Omega^D M\big)$ so that $\int_V Q(t)$ is the amount in $V \subseteq M$.
Flow rate (flux $(D-1)$–form): $q \in C^\infty\big(I, \Omega^{D-1} M\big)$, so that $\int_S q(t)$ is net flow through S^{D-1}.
Internal production (source D–form): $f \in C^\infty\big(I, \Omega^D M\big)$, so that $\int_V f(t)$ is total production in V.

Flux Across an Internal Interface



Time–Integrated Net Flux Through S

For $[t_1, t_2] \subset I$,

$$\text{Net flux on } [t_1, t_2]: \qquad \int_{t_1}^{t_2} \Big(\int_S q(t) \Big)\, dt.$$

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Continuity Law: Integral and Differential Forms

Integral Balance on Any Subregion $V \subseteq M$

For $[t_1, t_2] \subset I$,

$$\underbrace{\int_V Q(t_2) - \int_V Q(t_1)}_{\text{amount change in } V} = \underbrace{\int_{t_1}^{t_2} \Big(\int_V f(t) \Big) \, dt}_{\text{internal production}} - \underbrace{\int_{t_1}^{t_2} \Big(\int_{\partial V} q(t) \Big) \, dt}_{\text{total outflow through } \partial V} .$$

Using Stokes–Cartan on ∂V,

$$\int_{\partial V} q(t) = \int_V d_{D-1} q(t).$$

Continuity Law

Integral Balance on Any Subregion $V \subseteq M$

Assuming smoothness, differentiate under the integral in time:

$$\int_V \frac{\partial Q}{\partial t}(t) = \int_V \big(f(t) - d_{D-1}q(t)\big).$$

Localization ("Dropping the Integrals")

Because the equality holds for *all* subregions V and for all t, the integrands must agree pointwise:

$$\boxed{\frac{\partial Q}{\partial t} = f - d_{D-1}q} \qquad \text{on } M \times I.$$

This is the differential–form version of the continuity law.

Boundary Sign Convention and Stokes on Moving Slices

Orientation and Signs

With the outward orientation on ∂V,

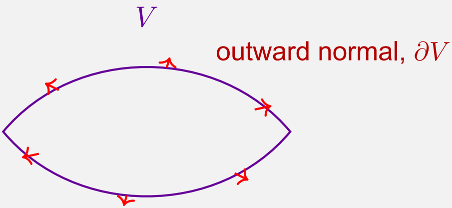
$$\int_{\partial V} q = \int_V d_{D-1} q \quad \Longrightarrow \quad \text{outflow (through } \partial V) = \int_V d_{D-1} q.$$

Thus,

$$\frac{\partial Q}{\partial t} = f - d_{D-1} q$$

encodes "time rate of change = production − outflow".

Visualization



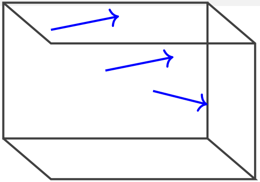
Correspondence with Vector Calculus

A Common Identification

In $D = 3$, write $Q = \tilde{\rho}\, dV$ (amount density $\tilde{\rho}$ times 3–volume form), and represent a vector flux field $\tilde{F}$ via the flux 2–form $q = \iota_{\tilde{F}}\mathrm{vol}$. Then

$$\frac{\partial Q}{\partial t} = f - d_2 q \quad \Longleftrightarrow \quad \frac{\partial \tilde{\rho}}{\partial t}\, dV = \tilde{f}\, dV - (\nabla \cdot \tilde{F})\, dV$$

$$\Longleftrightarrow \quad \boxed{\frac{\partial \tilde{\rho}}{\partial t} = \tilde{f} - \nabla \cdot \tilde{F}}\,.$$



flux field $\tilde{F}$, $q = \iota_{\tilde{F}}\mathrm{vol}$

Neumann Boundary and Prescribed Flux

Boundary Portion and Data

Let $\Gamma_N \subseteq \partial M$ be a $(D-1)$–dimensional smooth submanifold (Neumann boundary). A *prescribed* flux (i.e. *given as boundary input*) is a $(D-1)$–form

$$g_N \in C^\infty\big(I, \Omega^{D-1}\Gamma_N\big) \quad \text{with physical unit } [X\,T^{-1}],$$

interpreted so that $\int_S g_N(t)$ equals the net amount per unit time crossing $S \subset \Gamma_N$ at time t.

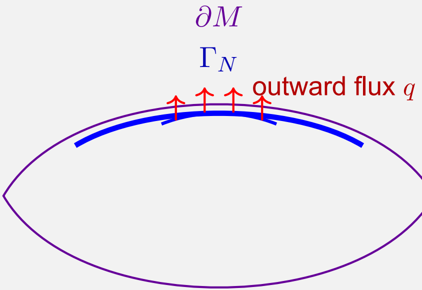
Neumann Boundary Condition (NBC)

The total flux $q \in C^\infty\big(I, \Omega^{D-1}M\big)$ satisfies

$$\mathrm{tr}_{\Gamma_N}\, q \;=\; g_N \qquad \text{on } \Gamma_N \times I.$$

Neumann Boundary and Prescribed Flux

Boundary Portion and Data



Potential, Diffusion, and Advection

Potential and Total Flux Split

Let $u \in C^\infty(I, \Omega^0 M)$ be the *potential* (units $[Y]$), e.g. temperature, concentration, pressure, or electric potential. Split the total flux as

$$q = q_D + q_A,$$

where q_D is the *diffusive* part and q_A is the *advective* part.

Diffusive (constitutive) law — isotropic & anisotropic

With $du \in \Omega^1 M$ and Hodge star $\star : \Omega^1 \to \Omega^{D-1}$,

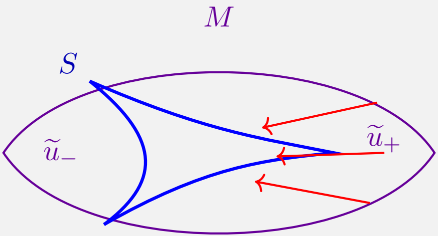
$$\begin{aligned} \text{(isotropic)} \quad & q_D = -\kappa \star (du), \quad \kappa \in C^\infty(M),\ \kappa > 0; \\ \text{(anisotropic)} \quad & q_D = -\star\big(\widetilde{\kappa}(du)\big), \quad \widetilde{\kappa} : \Omega^1 M \to \Omega^1 M \\ & \text{(SPD } (1,1)\text{–tensor).} \end{aligned}$$

Potential, Diffusion, and Advection

Diffusive (constitutive) law — isotropic & anisotropic

Equivalently, define $\kappa : \Omega^{D-1}M \to \Omega^{D-1}M$ by the identity

$$\boxed{\star_{D-1} \circ \widetilde{\kappa} \;=\; \kappa \circ \star_1} \qquad \Rightarrow \qquad q_D \;=\; -\kappa \star (du).$$



$$q|_S \sim \widetilde{u}_+ - \widetilde{u}_-$$

Advective Flux via Volume Flux Form

Let $\tilde{v}$ be a velocity vector field; its *volume flux form* is

$$v \ := \ \iota_{\tilde{v}} \mathrm{vol} \ \in \ \Omega^{D-1}M \qquad \text{(units } [L^D T^{-1}]).$$

If $Q \in \Omega^D M$ is the amount D–form, then

$$\boxed{q_A \ = \ (\star Q)\, v} \qquad \text{(scalar density } \star Q \ \times \ \text{volume flux form } v) \,.$$

Volumetric Capacity

Relate amount and potential by a capacity coefficient:

$$\tilde{\pi} : \Omega^0 M \to \Omega^0 M, \qquad Q = \star(\tilde{\pi}\, u).$$

Equivalently, $\exists\, \pi : \Omega^D M \to \Omega^D M$ with $\boxed{\pi \circ \star_0 = \star_0 \circ \tilde{\pi}}$,

$$\Rightarrow \quad Q = \pi(\star u).$$

Hence

$$\frac{\partial Q}{\partial t} = \pi\left(\star \frac{\partial u}{\partial t}\right) = \star\left(\tilde{\pi}\, \frac{\partial u}{\partial t}\right).$$

Boundary Partition and Data

Dirichlet vs. Neumann

Let $\Gamma_D, \Gamma_N \subseteq \partial M$ be relatively open, disjoint, with

$$\Gamma_D \cup \Gamma_N = \partial M, \qquad \text{dim}(\Gamma_D \cap \Gamma_N) \leq D - 2.$$

Dirichlet boundary condition (DBC):

$$\text{tr}_{\Gamma_D} u \ = \ u_D, \qquad u_D \in C^\infty(I, \Omega^0 \Gamma_D).$$

Neumann boundary condition (NBC):

$$\text{tr}_{\Gamma_N} q \ = \ g_N, \qquad g_N \in C^\infty(I, \Omega^{D-1} \Gamma_N).$$

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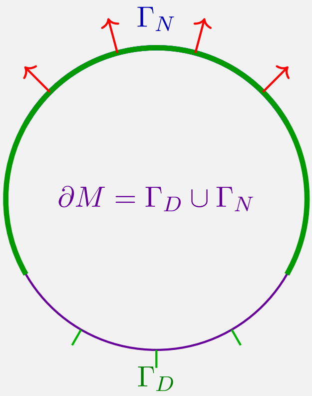
Visualization



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Governing Equations (Summary)

Bulk Law (Continuity)

$$\frac{\partial Q}{\partial t} = f - d_{D-1} q \qquad \text{in } M \times I.$$

Constitutive Relations

Capacity: $Q = \star(\widetilde{\pi}\, u) = \pi(\star u), \quad \pi \circ \star_0 = \star_0 \circ \widetilde{\pi}.$

Diffusion: $q_D = -\star\big(\widetilde{\kappa}(du)\big) = -\kappa \star (du), \quad \star_{D-1} \circ \widetilde{\kappa} = \kappa \circ \star_1.$

Advection: $q_A = (\star Q)\, v, \quad v = \iota_{\widetilde{v}} \mathrm{vol} \in \Omega^{D-1} M.$

Governing Equations (Summary)

Boundary Conditions and Unknowns

Unknowns: $u \in C^\infty(I, \Omega^0 M)$, $\quad Q \in C^\infty(I, \Omega^D M)$, $q \in C^\infty(I, \Omega^{D-1} M)$.

Dirichlet on Γ_D : $\quad \mathrm{tr}_{\Gamma_D} u = u_D$.

Neumann on Γ_N : $\quad \mathrm{tr}_{\Gamma_N} q = g_N$.

Total flux: $\quad q = q_D + q_A$.

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Primal Strong Model

Continuity and Constitutive Content (recall)

Continuity: $$\frac{\partial Q}{\partial t} = f - d_{D-1} q,$$

Amount–potential link (capacity): $Q = \star(\widetilde{\pi}\, u)$

$\big($equivalently $\widetilde{\pi}\, u = \star_D Q\big)$,

Flux split: $q = q_D + q_A,$

Diffusion (anisotropic): $q_D = - \star(\widetilde{\kappa}\, du)$

$\big($isotropic: $q_D = -\kappa \star du\big)$,

Advection: $q_A = (\star Q)\, v, \quad v = \iota_{\widetilde{v}}\mathrm{vol} \in \Omega^{D-1} M.$

Primal Strong Model

Substitute Q, q in terms of u

$$
\begin{aligned}
\frac{\partial}{\partial t}\Big(\star(\widetilde{\pi}\, u)\Big) &= f - d_{D-1}\Big(- \star(\widetilde{\kappa}\, du) \;+\; (\star Q)\, v\Big) \\
&= f \;+\; d_{D-1} \star (\widetilde{\kappa}\, du) \;-\; d_{D-1}\big((\star Q)\, v\big) \\
&= f \;+\; d_{D-1} \star (\widetilde{\kappa}\, du) \;-\; d_{D-1}\Big((\star \star (\widetilde{\pi} u))\, v\Big),
\end{aligned}
$$

where in the last line we used $Q = \star(\widetilde{\pi} u)$.

Primal Strong Model

Primal strong equation for u

$$\boxed{\frac{\partial}{\partial t}\Big(\star\,(\widetilde{\pi}u)\Big) \;-\; d_{D-1}\star(\widetilde{\kappa}\,du) \;+\; d_{D-1}\Big((\star\star(\widetilde{\pi}u))\,v\Big) \;=\; f}$$

In isotropic diffusion ($\widetilde{\kappa} = \kappa$ id), the second term is $d_{D-1}(\kappa \star du)$.

Equivalent Form Using $\widetilde{\pi} u = \star_D Q$

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Express everything through Q and then back to u

$$\widetilde{\pi} u = \star_D Q \quad \Longleftrightarrow \quad \star(\widetilde{\pi} u) = \star\star_D Q \;=\; \sigma_D\, Q,$$

where $\sigma_D = \pm 1$ depends on the metric signature and D (in Euclidean D–RM, $\sigma_D = +1$). Hence

$$\frac{\partial}{\partial t}\Big(\star\,(\widetilde{\pi} u)\Big) = \sigma_D\,\frac{\partial Q}{\partial t}.$$

Using the continuity equation $\frac{\partial Q}{\partial t} = f - d_{D-1} q$ and $q = -\star\,(\widetilde{\kappa} du) + (\star Q)v$, we get

$$\begin{aligned}
\sigma_D\,\frac{\partial Q}{\partial t} &= f - d_{D-1}\Big(-\star(\widetilde{\kappa} du) + (\star Q)v\Big)\\
&= f + d_{D-1}\star(\widetilde{\kappa} du) - d_{D-1}\big((\star Q)v\big).
\end{aligned}$$

Replacing Q by $\star(\widetilde{\pi} u)$ recovers the primal equation in u from the previous slide.

Manufactured Steady Problem

Steady assumptions and model

Let $\frac{\partial u}{\partial t} = 0$, hence $\frac{\partial Q}{\partial t} = 0$, and $v = 0$. The continuity law reduces to

$$0 = f - d_{D-1} q \quad \Longrightarrow \quad f = d_{D-1} q.$$

With pure diffusion $q = q_D = -\kappa \star du$ (isotropic, constant $\kappa > 0$),

$$f = d_{D-1}\big(-\kappa \star du\big) = -\kappa\, d_{D-1}\big(\star du\big).$$

Manufactured Steady Problem

Choose $M = [0,1]^3 \subset \mathbb{R}^3$, standard Euclidean metric

Let $u(x,y,z) = x^2 + y^2 + z^2$ and $\kappa = 2$ (constant).

$$du = 2x\,dx + 2y\,dy + 2z\,dz,$$

$$\star dx = dy \wedge dz, \quad \star dy = dz \wedge dx, \quad \star dz = dx \wedge dy,$$

$$\star du = 2x\,dy \wedge dz \; + \; 2y\,dz \wedge dx \; + \; 2z\,dx \wedge dy,$$

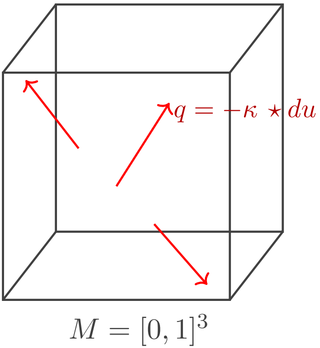
$$\begin{aligned} q \;=\; -\kappa \star du &= -2\Big(2x\,dy \wedge dz + 2y\,dz \wedge dx + 2z\,dx \wedge dy\Big) \\ &= -4x\,dy \wedge dz \; - \; 4y\,dz \wedge dx \; - \; 4z\,dx \wedge dy. \end{aligned}$$

Finally,

$$\begin{aligned} f = dq \;&=\; -\kappa\, d(\star du) = -2\,\big(\Delta u\big)\,dx \wedge dy \wedge dz = -2 \cdot 6\,dx \wedge dy \wedge dz \\ &= -12\,dx \wedge dy \wedge dz. \end{aligned}$$

Thus the manufactured source is the constant 3–form $f = -12\,dx \wedge dy \wedge dz$ on M.

Flux Sketch (steady diffusion in a cube)



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Summary

Main Concepts

- $\mu(M) = \int_M \text{vol}_{(M,g)}$ gives measure (length/area/volume) of (M, g).
- $\langle \omega, \eta \rangle_p = \int_M g_p^*(\omega, \eta)\text{vol}_{(M,g)}$ defines inner product on $\Omega^p(M)$.
- Codifferential d^* is the adjoint of d:

$$\langle d_p^* \omega, \eta \rangle_{p-1} = \langle \omega, d_{p-1}\eta \rangle_p.$$

- Relationship with Hodge star:

$$d_{D-p}^* \circ \star_p = (-1)^{p+1} \star_{p+1} \circ d_p.$$

- For $D = 3$: (d_0, d_1, d_2) correspond to $(\nabla, \nabla\times, \nabla\cdot)$ and (d_1^*, d_2^*) to their adjoints.

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Summary

Main Concepts

- Neumann data prescribes the *flux* $(D-1)$–form on Γ_N, Dirichlet data prescribes the *potential* on Γ_D.
- Diffusion: $q_D = -\star(\widetilde{\kappa}\, du)$ (or $-\kappa \star du$); Advection: $q_A = (\star Q)\, v$ with $v = \iota_{\widetilde{v}}\mathrm{vol}$.
- Capacity links amount and potential: $Q = \star(\widetilde{\pi} u)$.
- Continuity in forms: $\dfrac{\partial Q}{\partial t} = f - d_{D-1} q$, closed by the constitutive laws and boundary conditions.

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Summary

Main Concepts

- Amount $Q(t) \in \Omega^D M$, flux $q(t) \in \Omega^{D-1} M$, production $f(t) \in \Omega^D M$ model transport on (M, g).
- Integral balance on any $V \subseteq M$ and Stokes–Cartan yield the local continuity law

$$\frac{\partial Q}{\partial t} = f - d_{D-1} q.$$

- In $D = 3$, with $Q = \tilde{\rho}\, dV$ and $q = \iota_{\tilde{F}} \mathrm{vol}$, this becomes

$$\frac{\partial \tilde{\rho}}{\partial t} = \tilde{f} - \nabla \cdot \tilde{F}.$$

Summary

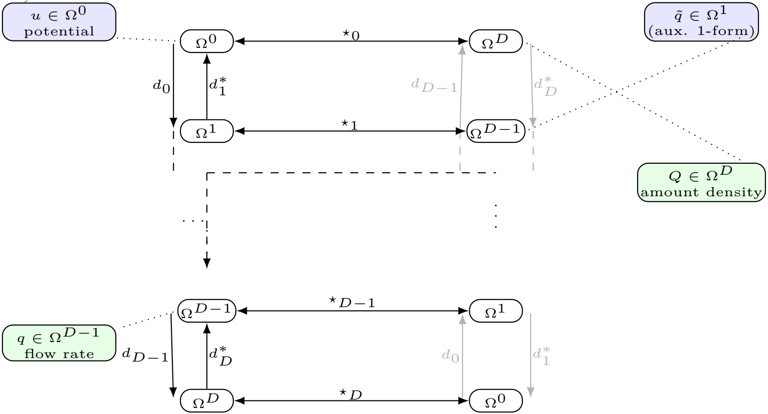
Main Concepts

- Primal strong model (exterior–calculus form):

$$\frac{\partial}{\partial t}\big(\star(\widetilde{\pi}u)\big) - d_{D-1}\star(\widetilde{\kappa}du) + d_{D-1}\big((\star\star(\widetilde{\pi}u))v\big) \;=\; f.$$

- Steady, no advection: $f = dq = -\kappa\, d(\star du)$; in $\mathbb{R}^3$ this is $f = -\kappa\,\Delta u$ vol.
- Manufactured 3D example with $u = x^2 + y^2 + z^2$, $\kappa = 2$:
  $q = -4x\,dy \wedge dz - 4y\,dz \wedge dx - 4z\,dx \wedge dy,$
  $f = -12\,dx \wedge dy \wedge dz$.

Global Picture



Codifferential: $d^*_{D-p} = (-1)^{p+1+p(D-p)} \star_{p+1} \circ d_p \circ \star_{D-p}$

Inner product: $g_p^*(\omega, \eta)\,\mathrm{vol} = \omega \wedge \star_p \eta$

Hodge involution:

$\star_{D-p} \circ \star_p = (-1)^{p(D-p)}\,\mathrm{id}_{\Omega^p}$

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Thanks