Combinatorial Mesh Calculus (CMC)

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1 Commutative rings with unity

Definition 1.1. Let R be a set, $0, 1 \in R$, $-\colon R \to R$ $+, *\colon R \times R \to R$. We say that $(R, 0, 1, -, +, *)$ is a **commutative ring with unity** (we will simply say that R is commutative ring with unity when operations are clear from the context) if the following equalities are satisfied for all $a, b, c \in R$:

$$a + (b + c) = (a + b) + c \quad \text{(addition is associative)}, \tag{1.1a}$$

$$a + 0 = 0 + a = a \quad \text{(0 is neutral with respect to addition)}, \tag{1.1b}$$

$$a + (-a) = (-a) + a = 0 \quad \text{(negation is the inverse of addition)}, \tag{1.1c}$$

$$a + b = b + a \quad \text{(addition is commutative)}, \tag{1.1d}$$

$$a * (b + c) = a * b + a * c \quad \text{(multiplication is left-distributive over addition)}, \tag{1.1e}$$

$$(a + b) * c = a * c + b * c \quad \text{(multiplication is right-distributive over addition)}, \tag{1.1f}$$

$$a * (b * c) = (a * b) * c \quad \text{(multiplication is associative)}, \tag{1.1g}$$

$$a * 1 = 1 * a = a \quad \text{(1 is neutral with respect to multiplication)}, \tag{1.1h}$$

$$a * b = b * a \quad \text{(multiplication is commutative)}. \tag{1.1i}$$

We will often omit multiplication sign and write ab instead of $a * b$.

Example 1.2. The following are examples and counterexamples of commutative rings with unity.

1. The number sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ of respectively integers, rationals, reals, and complex numbers are commutative rings with unity with respect to the standard arithmetic operations. The set $\mathbb{N}$ of natural numbers is not a ring since it lacks negation.

2. For any positive integer n, the set $\mathbb{Z}_n$ of integers modulo n is a commutative ring with unity with respect to addition and multiplication modulo n. (This is the original inspiration behind the word "ring" as its elements can be arranged in a loop, like the hours on a clock.)

3. Let X be a set, R be a commmutative ring with unity. Then the set of functions $R^X := \{f \mid f\colon X \to R\}$ is a commutative ring with unity ring with respect to pointwise operations: if $f, g\colon X \to R$, $x \in X$, we define

$$0_{R^X}(x) := 0_R, \tag{1.2a}$$

$$1_{R^X}(x) := 1_R, \tag{1.2b}$$

$$(-_{R^X} f)(x) := -_R (f(x)), \tag{1.2c}$$

$$(f +_{R^X} g)(x) := f(x) +_R g(x), \tag{1.2d}$$

$$(f *_{R^X} g)(x) := f(x) *_R g(x). \tag{1.2e}$$

We will drop subscripts and overload ring operations on R^X.

Definition 1.3. Let R be a commutative ring with unity, S be a subset of R. We say that S is a **subring with unity** of R if S is a commutative ring with unity with operations on S being the restrictions on S of the operations in R.

Proposition 1.4. Let R be a commutative ring with unity, S be a subset of R. Then S is a subring with unity if contains 1 and is closed under negation, addition, and multiplication, i.e., for any $a, b \in R$,

$$1 \in R, \; -a \in R, \; a + b \in R, \; ab \in R. \tag{1.3}$$

Example 1.5. The following are examples of subrings.

1. The number sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ form a chain of subrings (any of them is a subring with unity of the next one).
2. For any $n \in \mathbb{N}$, $n \geq 2$, the set $n\mathbb{Z} := \{nx \mid x \in \mathbb{Z}\}$ is a subring of $\mathbb{Z}$ that has no unity.

Definition 1.6. Let R be a commutative ring with unity. We say that R is a **field** if it has at least two elements and any nonzero element has a multiplicative inverse, i.e.,

$$\forall a \in R \setminus \{0\}\ \exists b \in R,\ a * b = 1. \tag{1.4}$$

Example 1.7. The following are examples and counterexamples of fields.

1. Among the number sets, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are fields, $\mathbb{Z}$ is not a field.
2. Let n be an integer. Then $\mathbb{Z}_n$ is a field if and only if n is a prime number.
3. Let X be a nonempty set, R be a field. Then the set of functions $R^X := \{f \mid f\colon X \to R\}$ is not a field except in the trivial case of X having one element. Indeed, let X has at least two elements, and take a function $f \in R^X$ that is zero at a point x_0 but nonzero at some other point x_1 (so it is not identically zero). Then f is not invertible since for any $g \in R^X$,

$$(f * g)(x_0) = f(x_0) * g(x_0) = 0 * g(x_0) = 0 \neq 1. \tag{1.5}$$

Remark 1.8. The reason for introducing commutative rings with unity instead of working with fields is that, as explained above, sets of functions R^X are not fields. We will encounter them a lot when working with algebraic structures on smooth manifolds and meshes, predominantly modules over subrings of R^X.

2 Modules over commutative rings with unity

Definition 2.1. Let $(R, 0_R, 1_R, -_R, +_R, *_R)$ be a commutative ring with unity, V be a set, $0 \in V$, $-\colon V \to V$, $+\colon V \times V \to V$, $*\colon R \times V \to V$. We say that $(V, 0, -, +, *)$ is a **module** over R if the following equalities are satisfied for all $u, v, w \in V$, $\lambda, \mu \in R$:

$$u + (v + w) = (u + v) + w \quad \text{(addition is associative)}, \tag{2.1a}$$
$$v + 0 = 0 + v = v \quad \text{(0 is neutral with respect to addition)}, \tag{2.1b}$$
$$v + (-v) = (-v) + v = 0 \quad \text{(negation is the inverse of addition)}, \tag{2.1c}$$
$$u + v = v + u \quad \text{(addition is commutative)}, \tag{2.1d}$$
$$\lambda * (u + v) = \lambda * u + \lambda * v \quad \text{(scalar multiplication is distributive over vector addition)}, \tag{2.1e}$$
$$(\lambda +_R \mu) * v = \lambda * v + \mu * v \quad \text{(scalar multiplication is distributive over scalar addition)}, \tag{2.1f}$$
$$\lambda * (\mu * v) = (\lambda *_R \mu) * v \quad \text{(scalar multiplication is ``associative'')}, \tag{2.1g}$$
$$1_R * v = v \quad \text{(1_R is neutral with respect to scalar multiplication)}. \tag{2.1h}$$

We will often omit multiplication sign and write λv instead of $\lambda * v$. The elements of V are called *vectors*, while those of R are called *scalars*.

Remark 2.2. When the space of scalars is a field, a module becomes a vector space. Hence, modules generalise vector spaces.

Example 2.3. Let R be a commutative ring with unity. The following are examples of modules over R.

1. For any $n \in \mathbb{N}$, the space R^n is a module over R under component-wise addition and multiplication with a scalar.
2. For any $m, n \in \mathbb{N}$, the space $M_{m \times n}(R)$ of $m \times n$ matrices with elements in R is a module over R under under component-wise addition and multiplication with a scalar.
3. For any set X, the ring R^X can also be considered as a module over R with pointwise addition and multiplication with a scalar. It generalises the previous two cases when $X = \{1, ..., n\}$ and $X = \{1, ..., m\} \times \{1, ..., n\}$ respectively.

Definition 2.4. Let R be a commutative ring with unity, V be a module over R. $S \subset V$. We say that the S is **linearly independent** if for any $n \in \mathbb{N}$, $\lambda_1, ..., \lambda_n \in R$, $v_1, ..., v_n \in V$,

$$\lambda_1 v_1 + ... + \lambda_n v_n = 0 \Rightarrow \lambda_1 = ... = \lambda_n = 0. \tag{2.2}$$

We say that the set S is **linearly dependent** if there exist $n \in \mathbb{N}$, $\lambda_1, ..., \lambda_n \in R \setminus \{0\}$, $v_1, ..., v_n \in V$ such that

$$\lambda_1 v_1 + ... + \lambda_n v_n = 0. \tag{2.3}$$

Definition 2.5. Let R be a commutative ring with unity, V be a module over R. $S \subset V$. We say that S **spans** V if for any $v \in V$ there exist $n \in \mathbb{N}$, $\lambda_1, ..., \lambda_n \in R$, $v_1, ..., v_n \in V$ such that

$$v = \lambda_1 v_1 + ... + \lambda_n v_n. \tag{2.4}$$

Definition 2.6. Let R be a commutative ring with unity, V be a module over R. $S \subset V$. We say that S is a **basis** of V if it is linearly independent and spans V.

Definition 2.7. Let R be a commutative ring with unity, V be a module over R. We say that V is **free** if it admits a basis.

Example 2.8. The module of tuples and matrices over a commutative ring with unity are free. A set of modules that do not have a basis consists of $\mathbb{Z}_n$ as modules over $\mathbb{Z}$ for any $n \geq 2$. Indeed, any vector $v \in \mathbb{Z}_n$ is linearly independent since $nv = 0$.

3 Algebras over rings

Definition 3.1. Let R be a commutative ring with unity, A be an R-module, $\mu \colon R \times R \to R$. We say that (A, μ) is an R-**algebra** (or **algebra over** R) if for all $a, b, c \in A$, $\lambda \in R$:

$$\mu(a+b, c) = \mu(a, c) + \mu(b, c) \text{(multiplication on } A \text{ is left-distributive)}, \tag{3.1a}$$
$$\mu(a, b+c) = \mu(a, b) + \mu(a, c) \text{(multiplication on } A \text{ is right-distributive)}, \tag{3.1b}$$
$$\mu(\lambda * a, b) = \lambda * \mu(a, b) \qquad (\mu \text{ respects multiplication of scalar on the left}), \tag{3.1c}$$
$$\mu(a, \lambda * b) = \lambda * \mu(a, b) \qquad (\mu \text{ respects multiplication of scalar on the right}). \tag{3.1d}$$

Definition 3.2. Let R be a commutative ring with unity, (A, μ) be an R-algebra.

1. We say that (A, μ) is **associative algebra** if for any $a, b, c \in A$, $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$. In other words, (A, μ) is a ring.
2. We say that (A, μ) is **commutative algebra** if for any $a, b, c \in A$, $\mu(a, b) = \mu(b, a)$.
3. Let $1 \in R$. We say that $(A, \mu, 1)$ is **unital algebra** if for any $a, \in A$, $\mu(a, 1) = \mu(1, a) = a$. (1 is the **unit element** or **unity**.)
4. We say that (A, μ) is **alternating algebra** if for any $a \in A$, $\mu(a, a) = 0$.
5. We say that (A, μ) is **anti-commutative algebra** if for any $a, b \in A$, $\mu(a, b) = -\mu(b, a)$.

Proposition 3.3. Let R be a commutative ring with unity, Let (A, μ) be an algebra.

- If A is alternating, then A is anti-commutative.
- If A is anti-commutative and the ring R has the property

$$\forall x \in R, \; x + x = 0 \Rightarrow x = 0, \tag{3.2}$$

  then A is alternating.

Proof. First, let A be alternating. Then for any $a, b \in A$,

$$0 = \mu(a+b, a+b) = \mu(a, a) + \mu(a, b) + \mu(b, a) + \mu(b, b) = \mu(a, b) + \mu(b, a) \Rightarrow \mu(b, a) = -\mu(a, b). \tag{3.3}$$

Conversely, let A be anti-commutative. Taking $b = a$ in the definition leads to

$$\mu(a,a) = -\mu(a,a) \Rightarrow \mu(a,a) + \mu(a,a) = 0. \tag{3.4}$$

Under the assumption of Equation (3.2), we conclude that $\mu(a,a) = 0$. □

4 Lie algebras

Definition 4.1. Let R be a commutative ring with unity, V be an R-algebra with multiplication operation $[\cdot,\cdot]$. We say that V is a **Lie algebra** if $[\cdot,\cdot]$ is alternating and satisfies the **Jacobi identity**: for any $x, y, z \in V$,

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0. \tag{4.1}$$

Proposition 4.2. Let R be a commutative ring with unity, V be an associative algebra over R. Define the **commutator**$[\cdot,\cdot]\colon V \times V \to V$ as follows: for any $x, y \in V$,

$$[x,y] := xy - yx. \tag{4.2}$$

Then $(V, [\cdot,\cdot])$ is a Lie algebra. In particular, when X is a module over R, $V := \mathrm{End}_R X$ is a Lie algebra with Lie bracket given by

$$[\varphi,\psi] := \varphi \circ \psi - \psi \circ \varphi, \ \ \varphi, \psi \in \mathrm{End}_R X. \tag{4.3}$$

Proof. Let $x, y, z \in A$. $[\cdot,\cdot]$ is alternating since $[x,x] = xx - xx = 0$. Next, we calculate

$$[x,[y,z]] = [x, yz - zy] = x(yz - zy) - (yz - zy)x = xyz - xzy - yzx + zyx. \tag{4.4}$$

Similarly, $[y,[z,x]] = yzx - yxz - zxy + xzy$ and $[z,[x,y]] = zxy - zyx - xyz + yxz$. We can see that each of the six permutations of (x,y,z) in products occurs twice with opposite signs, which means that their sum is zero. Hence, the Jacobi identity is also satisfied. □

Definition 4.3. Let R be a commutative ring with unity, $(V, [\cdot,\cdot]_V)$ and $(W, [\cdot,\cdot]_W)$ be Lie algebras over R, $\varphi \in \mathrm{Hom}_R(V,W)$. We say that φ is a **Lie algebra homomorphism** if for any $u, v \in V$,

$$\varphi([u,v]_V) = [\varphi u, \varphi v]_W. \tag{4.5}$$

Definition 4.4. R be a commutative ring with unity, V be a Lie algebra over R. Define the **adjoint map**

$$\mathrm{adj}\colon V \to \mathrm{End}_R V, \ \ \mathrm{adj}_x y := [x,y], \ \ x, y \in V. \tag{4.6}$$

Proposition 4.5. R be a commutative ring with unity, V be a Lie algebra over R. Then the adjoint map is a homorphism of Lie algebras, i.e., for any $x, y \in V$,

$$\mathrm{adj}_{[x,y]} = [\mathrm{adj}_x, \mathrm{adj}_y] := \mathrm{adj}_x \circ \mathrm{adj}_y - \mathrm{adj}_y \circ \mathrm{adj}_x. \tag{4.7}$$

Proof. Let $x, y, z \in V$. Then

$$\begin{aligned}[\mathrm{adj}_x, \mathrm{adj}_y]z &= (\mathrm{adj}_x \circ \mathrm{adj}_y - \mathrm{adj}_y \circ \mathrm{adj}_x)z = [x,[y,z]] - [y,[x,z]] \\ &= [x,[y,z]] + [y,[z,x]] = -[z,[x,y]] = [[x,y],z] = \mathrm{adj}_{[x,y]} z. \quad \square\end{aligned} \tag{4.8}$$

5 Derivations on algebras

Definition 5.1. Let R be a commutative ring with unity, A be an R-algebra, $D \in \mathrm{Hom}_R(A,A)$. We say that D is a **derivation** if for any $a, b \in A$ the **Leibniz rule** (or the **product rule**) holds:

$$D(a * b) = (Da) * b + a * (Db). \tag{5.1}$$

By $\mathrm{Der}_R(A)$ we will denote the set of all derivations on A over R.

Example 5.2 (Derivative of a single variable function is derivation)**.** Let $R = \mathbb{R}$, $A = C^\infty(\mathbb{R})$ be the space of infinitely differentiable functions, D be the derivative operator, i.e., $Df = f'$ for $f \in A$. Then D is a derivation.

Example 5.3. Let R be a commutative ring with unity, V be a Lie algebra over R. Then the adjoint homomorphism $\mathrm{adj}\colon V \to \mathrm{End}_R(V)$ induces for each $x \in V$ a derivation $\mathrm{adj}_x\colon V \to V$. Indeed, take $x, y, z \in A$. Then

$$\mathrm{adj}_x[y,z] = [x,[y,z]] = -[z,[x,y]] - [y,[z,x]] = [[x,y],z] + [y,[x,z]] = [\mathrm{adj}_x y, z] + [y, \mathrm{adj}_x z]. \tag{5.2}$$

Proposition 5.4. Let R be a commutative ring with unity, A be a unital, commutative, and associative R-algebra. Then $\mathrm{Der}_R(A)$ is an A-module under pointwise addition and multiplication with a scalar.

Proof. Let $D, D_1, D_2 \in \mathrm{Der}_R(A)$. Then $D_1 + D_2$ is a homomorphism from module theory. To prove it satisfies the Leibniz rule, let $a, b \in A$. Then

$$(D_1+D_2)(a*b) = D_1(a*b)+D_2(a*b) = D_1a*b+a*D_1b+D_2a*b+a*D_2b = (D_1+D_2)a*b+a*(D_1+D_2)b. \tag{5.3}$$

Further, let $a \in A$. Then $(a \cdot D)(b) := a * Db$ for any $b \in B$. It is trivial to check it it is a homomorphism. To check it is a derivation, let $b, c \in A$. Then

$$(a\cdot D)(b*c) = a*D(b*c) = a*(Db*c+b*Dc) = (a*Db)*c+b*(a*Dc) = (a\cdot D)b*c+b*(a\cdot D)c. \tag{5.4}$$

Hence, $\mathrm{Der}_R(A)$ is an A-module. □

Proposition 5.5. Let R be a commutative ring with unity, A be a unital, commutative, and associative R-algebra, $X, Y \in \mathrm{Der}_R(A)$. Then $X \circ Y - Y \circ X \in \mathrm{Der}_R(A)$.

Proof. Linearity of $[X,Y]$ follows from module theory. Hence, we only need to show it satisfies the Leibniz rule. Let $f, g \in A$. Then

$$X(Y(fg)) = X((Yf)*g + f*(Yg)) = (X(Yf))*g + (Yf)*(Xg) + (Xf)*(Yg) + f*(X(Yg)). \tag{5.5}$$

Analogously,

$$Y(X(fg)) = (Y(Xf))*g + (Xf)*(Yg) + (Yf)*(Xg) + f*(Y(Xg)). \tag{5.6}$$

Hence,

$$[X,Y](fg) = (X(Yf))*g + f*(X(Yg)) - (Y(Xf))*g - f*(Y(Xg)) = ([X,Y]f)*g + f*([X,Y]g). \tag{5.7}$$

Corollary 5.6. Let R be a commutative ring with unity, A be a unital, commutative, and associative R-algebra. Then $\mathrm{Der}_R(A)$ is a Lie algebra with Lie bracket given by

$$[X,Y] := X \circ Y - Y \circ X, \;\; X, Y \in \mathrm{Der}_R(A). \tag{5.8}$$

Proposition 5.7. Let R be a commutative ring with unity, n be positive integer. Consider the set R^n as a ring with pointwise addition and multiplication, and as an R-algebra with pointwise scalar multiplication. Then

$$\mathrm{Der}_R(R^n) = 0. \tag{5.9}$$

Proof. Let $D \in \mathrm{Der}_R(R^n)$. Consider the standard basis $(e_1, ..., e_n)$ of R^n and let the matrix of D in the basis be A, i.e., for $1 \le i, j \le n$,

$$De_i = \sum_{j=1}^n A_{j,i} e_j. \tag{5.10}$$

Let $1 \le i \le n$. Then, since mltiplication in R^n is commutative,

$$De_i = D(e_i * e_i) = De_i * e_i + e_i * De_i = 2De_i * e_i, \tag{5.11}$$

which translates to

$$\sum_{j=1}^n A_{j,i} e_j = 2\sum_{j=1}^n A_{j,i} e_j * e_i = 2A_{i,i} e_i. \tag{5.12}$$

For $j \neq i$ this leads to $A_{j,i} = 0$, while for $j = i$ it leads to $A_{i,i} = 2A_{i,i} \Rightarrow A_{i,i} = 0$. Therefore, for all $i, j \in \{1, ..., n\}$, $A_{i,j} = 0$. Hence, $A = 0$, which means that $D = 0$. □

6 Exterior algebra

Definition 6.1. Let R be a commutative ring with unity, V be a module over R. The **exterior algebra** of V, $\Lambda^\bullet V$ is the smallest associative algebra with unity containing V as a subspace and satisfying the *alternating rule*: for every $v \in V$, if $\wedge$ is the multiplication on $\Lambda^\bullet V$, then

$$v \wedge v = 0. \tag{6.1}$$

Proposition 6.2. Let R be a commutative ring with unity, V be a module over R, $v, w \in V$. Then on $\Lambda^\bullet V$,

$$w \wedge v = -v \wedge w. \tag{6.2}$$

Proof. By the alternating rule,

$$0 = (v + w) \wedge (v + w) = v \wedge v + v \wedge w + w \wedge v + w \wedge w = 0 + v \wedge w + w \wedge v + 0 \Rightarrow w \wedge v = -v \wedge w. \tag{6.3}$$

□

Proposition 6.3. Let R be a commutative ring with unity, $D \in \mathbb{N}$, V be a free module over R of dimension D, $e_0, ..., e_{D-1}$ be a basis of V. Then the 2^D-element set $S(e)$,

$$S(e) := \left\{e_{i_0} \wedge ... \wedge e_{i_{p-1}} \mid p \in \{0, ..., D\},\ 0 \leq i_0 < ... < i_{p-1} \leq D - 1\right\} \tag{6.4}$$

(for $p = 0$ the empty wedge product is defined to be 1) forms a basis of $\Lambda^\bullet V$.

Remark 6.4. Let R be a commutative ring with unity, $D \in \mathbb{N}$, V be a free module over R of dimension D, $e_0, ..., e_{D-1}$ be a basis of V. The elements of $S(e)$ which are wedge products of p vectors are called **p-vectors**. Obviously, there are $\binom{D}{p}$ of them. Denote by $\Lambda^p V$ the space spanned by those p vectors (it does not depend on the basis of e – in fact $\Lambda^p V$ is the space spanned by linear combinations of wedge products of arbitrary vectors). Then we have the module decomposition

$$\Lambda^\bullet V := \sum_{p=0}^{D} \Lambda^p V. \tag{6.5}$$

This decompositions turns $\Lambda^\bullet V$ into a graded algebra, that is for each $p, q \in \mathbb{N}$,

$$\omega_p \in \Lambda^p V,\ \eta_q \in \Lambda^q V \Rightarrow \omega_p \wedge \eta_q \in \Lambda^{p+q} V \tag{6.6}$$

(here we define $\Lambda^r = 0$ for $r > D$).

Proposition 6.5. Let R be a commutative ring with unity, $D \in \mathbb{N}$, V be a free module over R of dimension D, $v_0, ..., v_{D-1} \in V$. Then $(v_0, ..., v_{D-1})$ form a basis of V if and only if

$$v_0 \wedge ... \wedge v_{D-1} \tag{6.7}$$

forms a basis of the 1-dimensional module $\Lambda^D V$.

Proposition 6.6. Let R be a commutative ring with unity, $D \in \mathbb{N}$, V be a free module over R of dimension D, $e_0, ..., e_{D-1}$ be a basis of V, $v_0, ..., v_{D-1} \in V$ such that for $j = 0, ..., D - 1$,

$$v_j = \sum_{i=0}^{D-1} e_i A_{i,j} \tag{6.8}$$

(in matrix form, $(v_0, ..., v_{D-1}) = (e_0, ..., e_{D-1})A$). Then

$$v_0 \wedge ... \wedge v_{D-1} = (\det A) e_0 \wedge ... \wedge e_{D-1}. \tag{6.9}$$

As a consequence, the matrix A is a change of basis matrix if and only if $\det A$ is an invertible element of R (a nonzero element when R is a field).

Notation 6.7. Let R be a commutative ring with unity and V be a module over R. Denote by V^* the **dual space** of V:

$$V^* := \mathrm{Hom}(V, R). \tag{6.10}$$

Proposition 6.8. Let R be a commutative ring with unity, V be a a finite-dimensional R-module, $p \in \mathbb{N}$. Then the R-modules $(\Lambda^p V)^*$ and $\Lambda^p(V^*)$ are canonically isomorphic. The isomorphism is the unique linear map $f\colon \Lambda^p(V^*) \to (\Lambda^p V)^*$ such that for any $\omega_0, ..., \omega_{p-1} \in V^*$ and any $v_0, ..., v_{p-1} \in V$,

$$f(\omega_0 \wedge ... \wedge \omega_{p-1})(v_0 \wedge ... \wedge v_{p-1}) = \det(\omega_i(v_j))_{i,j=0}^{p-1}. \tag{6.11}$$

Corollary 6.9. Let R be a commutative ring with unity, V be a a finite-dimensional R-module. Then $(\Lambda^\bullet V)^* \equiv \Lambda^\bullet(V^*)$.

Notation 6.10. Let V be an 1-dimensional real vector space, $v \in V$, $w \in V \setminus \{0\}$. By v/w denote the unique $\lambda \in \mathbb{R}$ such that $v = \lambda w$. (The uniqueness follows from the fact that the nonzero vector w forms a basis of V.)

Definition 6.11. Let $D \in \mathbb{N}$, V be a real vector space of dimension D, $v_0, ..., v_{D-1} \in V$. Consider the 1-dimensional space $\Lambda^D V$. Define an equivalence relation on the nonzero elements of $\Lambda^D V$ as follows: for any $v, w \in \Lambda^D V \setminus \{0\}$,

$$v \equiv w \Leftrightarrow v/w > 0. \tag{6.12}$$

This equivalence relation partitions $\Lambda^D V \setminus \{0\}$ into two equivalence classes (corresponding to positive and negative elements with respect to a choice). An **orientation on** V is a choice of 1 of the equivalence classes on $\Lambda^D V$. Equivalently, we will also specify orientation by choosing an element on $\Lambda^D(V^*) \equiv (\Lambda^D V)^*$.
An **oriented vector space** is a vector space with a chosen orientation.

Notation 6.12. Let $D \in \mathbb{N}$, $p \in \{0, ..., D\}$. By C_p^D we will denote the set of all ordered lists with p elements without repetition whose elements are from the set $\{0, ..., D-1\}$.

Notation 6.13. Let $D \in \mathbb{N}$, R be a commutative ring with unity, V be a D-dimensional R-moDule, $p \in \{0, ..., d\}$, $I_p := (i_0, ..., i_{p-1}) \in C_p^D$, $v_0, ..., v_{D-1} \in V$. By v_{I_p} we will denote the p-vector

$$v_{I_p} := v_{i_0} \wedge ... \wedge v_{i_{p-1}}. \tag{6.13}$$

Proposition 6.14. Let $D \in \mathbb{N}$, R be a commutative ring with unity, V be a D-dimensional R-module, $e_\bullet := (e_0, ..., e_{D-1})$ be a basis of V, $v_\bullet := (v_0, ..., v_{D-1})$ be a set of vectors, $a \in M_{D \times D}(\mathbb{R})$ be the transformation matrix from $e_\bullet$ to $v_\bullet$, $p \in \{0, ..., D\}$, $I_p \in C_p^D$. Then

$$v_{I_p} = \sum_{J_p \in C_p^D} \det\left(a|_{I_p \times J_p}\right) e_{J_p}. \tag{6.14}$$

7 Inner products and Hodge star

Definition 7.1. Let V be a vector space over $\mathbb{R}$. An **inner product** on V is a function $g\colon V \times V \to \mathbb{R}$ that is:

1. bilinear, i.e., for all $\lambda, \mu \in \mathbb{R}$, $u, v, w \in V$:

$$g(\lambda u + \mu v, w) = \lambda g(u, w) + \mu g(v, w), \; g(w, \lambda u + \mu v) = \lambda g(w, u) + \mu g(w, v); \tag{7.1}$$

2. symmetric, i.e., for all $u, v \in V$,

$$g(u, v) = g(v, u); \tag{7.2}$$

3. positive definite: for all $v \in V \setminus \{0\}$,

$$g(v, v) > 0. \tag{7.3}$$

If g is an inner product on V, the pair (V, g) is called an **real inner product space**.

Proposition 7.2. Let (V, g) be a finite-dimensional real inner product space. Define the map $\tilde{g}: V \to V^*$ as follows: for any $v \in V$,

$$\tilde{g}(v) := (w \in V \mapsto g(v, w)). \tag{7.4}$$

Then $\tilde{g}$ is an isomorphism.

Definition 7.3. Let $D \in \mathbb{N}$, (V, g) be a real inner product space of dimension D, $e = \{e_0, ..., e_{D-1}\}$ be a basis of V. We say that e is an **orthogonal basis** if any two disjoint elements are orthogonal, i.e., for all $i, j \in \{0, ..., D-1\}$, if $i \neq j$, then

$$g(e_i, e_j) = 0. \tag{7.5}$$

The basis e is called **orthonormal**, if it is orthogonal and for any $i \in \{0, ..., D-1\}$,

$$g(e_i, e_i) = 1. \tag{7.6}$$

An equivalent way of saying that e is orthonormal is by using the Kronecker delta symbol: for all $i, j \in \{0, ..., D-1\}$,

$$g(e_i, e_j) = \delta_{i,j} := \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}. \tag{7.7}$$

Definition 7.4. Let $d \in \mathbb{N}$, (V, g) be a real inner product space of dimension d, $p \in \{0, ..., d\}$. Define the **inner product on p-vectors** $\Lambda^p g: \Lambda^p V \times \Lambda^p V \to \mathbb{R}$ as follows: for any $v_0, ..., v_{p-1},\ w_0, ..., w_{p-1} \in V$,

$$(\Lambda^p g)(v_0 \wedge ... \wedge v_{p-1}, w_0 \wedge ... \wedge w_{p-1}) := \det(g(v_i, w_j))_{i,j=0}^{p-1} \tag{7.8}$$

(in other words, on simple p-vectors $\Lambda^p g$ is constructed using the Gram determinant). For arbitrary p-vectors, expand the above definition by bilinearity.
We define the **exterior algebra inner product**

$$\Lambda^\bullet g: \Lambda^\bullet V \times \Lambda^\bullet V \to \mathbb{R} \tag{7.9}$$

as follows: for $p, q \in \mathbb{N}$, $\omega_p \in \Lambda^p V$, $\eta_q \in \Lambda_q V$,

$$(\Lambda g)(\omega_p, \eta_q) := \begin{cases} (\Lambda^p g)(\omega_p, \eta_q), & p = q \\ 0, & p \neq q \end{cases}. \tag{7.10}$$

Definition 7.5. Let $d \in \mathbb{N}$, (V, g) be an oriented real inner product space of dimension d. The **volume d-vector** on V is the unique element of $\Lambda^d V$ which has norm 1 and is in the same oriented class as the chosen orientation.

Remark 7.6. On a non-oriented real inner product space V of dimension $d \in \mathbb{N}$ there are exactly two elements $\omega, \eta \in \Lambda^d V^*$ that have norm 1 (they are opposite to one another, i.e., $\omega = -\eta$) and so are candidates for a volume d-vector. Choosing one of them is equivalent to a choice of orientation on V.

Definition 7.7. Let $D \in \mathbb{N}$, (V, g) be an oriented real inner product space of dimension D, vol be the volume D-vector on $\Lambda^D V$, $p \in \mathbb{N}$. The **Hodge star operator on p-vectors** $\star_p$ is defined as the unique operator

$$\star_p: \Lambda^p V \to \Lambda^{D-p} V \tag{7.11}$$

such that for any $\omega \in \Lambda^p V$, $\eta \in \Lambda^{D-p} V$,

$$(\Lambda^{D-p} g)(\star_p \omega, \eta) \operatorname{vol} = \omega \wedge \eta. \tag{7.12}$$

The **Hodge star operator** $\star$ is the direct sum of all $\star_p$ for $p = 0, ..., D$.

Example 7.8. Consider $V = \mathbb{R}^2$ with the standard basis $e = (e_0, e_1)$, inner product g making e orthonormal, volume 2-vector $\mathrm{vol} := e_0 \wedge e_1$. Then

$$\star_0\, 1 = e_0 \wedge e_1 = \mathrm{vol}, \tag{7.13a}$$
$$\star_1\, e_0 = e_1, \tag{7.13b}$$
$$\star_1\, e_1 = -e_0, \tag{7.13c}$$
$$\star_2\, e_0 \wedge e_1 = \star_2\, \mathrm{vol} = 1. \tag{7.13d}$$

Proposition 7.9 (Alternative formula for Hodge star)**.** Let $D \in \mathbb{N}$, (V, g) be an oriented real inner product space of dimension D, vol be the volume D-vector on $\Lambda^D V$, $p \in \mathbb{N}$. Define the operator $\tilde{\star}_p$ as the unique operator

$$\tilde{\star}_p \colon \Lambda^p V \to \Lambda^{D-p} V \tag{7.14}$$

such that for any $\omega, \eta \in \Lambda^p V$,

$$\omega \wedge \tilde{\star}_p \eta = (\Lambda^p g)(\omega, \eta)\, \mathrm{vol}\,. \tag{7.15}$$

Then $\tilde{\star}_p = \star_p$.

Proposition 7.10. Let $D \in \mathbb{N}$ (V, g) be an oriented real inner product space of dimension D, $p \in \mathbb{N}$. Then

$$\star_{D-p} \circ \star_p = (-1)^{p(D-p)}\, \mathrm{id}_{\Lambda^p V}\,. \tag{7.16}$$

8 Chain complexes

Definition 8.1. Let R be a commutative ring with unity, $\{A_p\}_{p=-\infty}^{\infty}$ be a set of R-modules, $\{\partial_p \colon A_p \to A_{p-1}\}_{p=-\infty}^{\infty}$ be linear maps. Define $A_\bullet := \bigoplus_{p=-\infty}^{\infty} A_p$ and $\partial \colon A_\bullet \to A_\bullet$ to be the unique R-linear map such that for any $p \in \mathbb{Z}$ and any $a \in A_p$, $\partial a = \partial_p a$. We say that $(A_\bullet, \partial)$ is a **chain complex** if $\partial^2 = 0$. Equivalently, for any $p \in \mathbb{Z}$, $\partial_p \circ \partial_{p+1} = 0$.

Definition 8.2. Let $(A_\bullet, \partial^A)$ and $(B_\bullet, \partial^B)$ be chain complexes. Their **tensor product**

$$(C_\bullet, \partial^C) := (A_\bullet, \partial^A) \otimes (B_\bullet, \partial^B) \tag{8.1}$$

is defined as follows: for any $r \in \mathbb{Z}$,

$$C_r = \bigoplus_{p=-\infty}^{\infty} A_p \otimes B_{r-p}, \tag{8.2}$$

while the map ∂^C is defined for basis elements $a \otimes b \in A_p \otimes B_q \subseteq C_{p+q}$,

$$\partial^C(a \otimes b) := \partial^A a \otimes b + (-1)^p a \otimes \partial^B b. \tag{8.3}$$

Proposition 8.3. Let $(A_\bullet, \partial^A)$ and $(B_\bullet, \partial^B)$ be chain complexes,

$$(C_\bullet, \partial^C) = (A_\bullet, \partial^A) \otimes (B_\bullet, \partial^B) \tag{8.4}$$

be their tensor product (Definition 8.2). Then $(C_\bullet, \partial^C)$ is a chain complex.

9 Differential graded algebras

Definition 9.1. Let R be a commutative ring with unity, $\{A^p\}_{p=-\infty}^{\infty}$ be a set of R-modules, $\{\delta_p \colon A^p \to A^{p+1}\}_{p=-\infty}^{\infty}$ be linear maps, $\{\smile_{p,q} \colon A^p \times A^q \to A^{p+q}\}_{p,q=-\infty}^{\infty}$ be bilinear maps. Define $A^\bullet := \bigoplus_{p=-\infty}^{\infty} A^p$ and $\delta \colon A^\bullet \to A^\bullet$ to be the unique R-linear map such that for any $p \in \mathbb{Z}$ and any $\pi \in A^p$, $\delta\pi = \delta_p \pi$, $\smile \colon A^\bullet \times A^\bullet \to A^\bullet$ to be the unique R-bilinear map such that for any $p, q \in \mathbb{Z}$ and any $(\pi, \rho) \in A^p \times A^q$, $\pi \smile \rho = \pi \smile_{p,q} \rho$. We say that $(A^\bullet, \delta, \smile)$ is a **differential graded algebra** if $\delta^2 = 0$ and for any $p \in \mathbb{Z}$, $\pi \in A^p$, $\rho \in A^\bullet$,

$$\delta(\pi \smile \rho) = \delta\pi \smile \rho + (-1)^p \pi \smile \delta\rho. \tag{9.1}$$

Definition 9.2. Let $(A^\bullet, \delta^A, \smile^A)$ and $(B^\bullet, \delta^B, \smile^B)$ be differential graded algebras. Their **tensor product**

$$(C^\bullet, \delta^C, \smile^C) := (A^\bullet, \delta^A, \smile^A) \otimes (B^\bullet, \delta^B, \smile^B) \tag{9.2}$$

is defined as follows: for any $r \in \mathbb{Z}$,

$$C^r = \bigoplus_{p=-\infty}^{\infty} A^p \otimes B^{r-p}, \tag{9.3}$$

the map δ^C is defined for basis elements $a \otimes b \in A^p \otimes B^q \subseteq C^{p+q}$ as

$$\delta^C(a \otimes b) := \delta^A a \otimes b + (-1)^p a \otimes \delta^B b, \tag{9.4}$$

while the product is defined for basis elements $a \otimes b \in A^p \otimes B^q \subseteq C^{p+q}$ and $a' \otimes b' \in A^{p'} \otimes B^{q'} \subseteq C^{p'+q'}$ as

$$(a \otimes b) \smile^C (a' \otimes b') := (-1)^{qp'}(a \smile^A a') \otimes (b \smile^B b'). \tag{9.5}$$

Proposition 9.3. Let $(A^\bullet, \delta^A, \smile^A)$ and $(B^\bullet, \delta^B, \smile^B)$ be differential graded algebras,

$$(C^\bullet, \delta^C, \smile^C) := (A^\bullet, \delta^A, \smile^A) \otimes (B^\bullet, \delta^B, \smile^B) \tag{9.6}$$

be their tensor product (Definition 9.2). Then $(C^\bullet, \delta^C, \smile^C)$ is a differential graded algebra.

Part II
Smooth manifolds

10 Vector fields on manifolds

Definition 10.1. Let M be a smooth manifold. A **vector field** on M is a **derivation** on the $\mathbb{R}$-algebra $\mathcal{F}M$ of smooth functions. In other words, a vector field is an $\mathbb{R}$-linear map $X \colon \mathcal{F}M \to \mathcal{F}M$ such that for any $f, g \in \mathcal{F}M$ the **Leibniz rule** is satisfied:

$$X(fg) = f \cdot (Xg) + (Xf) \cdot g \tag{10.1}$$

(here $\cdot$ is multiplication of functions).
The space of all vector fields on M is denoted by $\mathfrak{X}M$. It is an $\mathbb{R}$-algebra under pointwise addition and multiplication with scalar. It is also an $(\mathcal{F}M)$-algebra with the following multiplication with a function: for any $f \in \mathfrak{X}M$ and $X \in \mathcal{F}M$, $fX \in \mathcal{F}M$ is defined to act on $g \in \mathcal{F}M$ by

$$(fX)g := f \cdot (Xg). \tag{10.2}$$

Proposition 10.2. Let M be a smooth manifold, $X, Y \in \mathfrak{X}M$. Then the **commutator**

$$[X, Y] := X \circ Y - Y \circ X \tag{10.3}$$

is also a vector field.

Proof. We will show that $[X, Y]$ satisfies the Leibniz rule. Let $f, g \in \mathcal{F}M$. Then

$$X(Y(fg)) = X(Yf \cdot g + f \cdot Yg) = X(Yf) \cdot g + Yf \cdot Xg + Xf \cdot Yg + f \cdot X(Yg). \tag{10.4}$$

Analogously,

$$Y(X(fg)) = Y(Xf) \cdot g + Xf \cdot Yg + Yf \cdot Xg + f \cdot Y(Xg). \tag{10.5}$$

Hence,

$$[X,Y](fg) = X(Y(fg)) - Y(X(fg)) = X(Yf) \cdot g + f \cdot X(Yg) - Y(Xf) \cdot g - f \cdot Y(Xg) = [X,Y]f \cdot g + f \cdot [X,Y]g. \tag{10.6}$$

Linearity of $[X, Y]$ is obvious. Hence, $[X, Y]$ is a vector field. □

Discussion 10.3. Let M be a smooth manifold of dimension D, $X, Y \in \mathfrak{X}M$, (U, φ) be a chart on M with local coordinates $x^1, ..., x^D$. Let X and Y be represented in local coordinates as

$$X = \sum_{p=1}^{D} f^p \frac{\partial}{\partial x^p}, \quad Y = \sum_{p=1}^{D} g^p \frac{\partial}{\partial x^p}. \tag{10.7}$$

We are going to derive expressions for $[X, Y]$. We have

$$X \circ Y = \sum_{p,q=1}^{D} f^p \frac{\partial}{\partial x^p} \circ (g^q \frac{\partial}{\partial x^q}) = \sum_{p,q=1}^{D} f^p \frac{\partial g^q}{\partial x^p} \frac{\partial}{\partial x^q} + \sum_{p,q=1}^{D} f^p g^q \frac{\partial^2}{\partial x^p \partial x^q}. \tag{10.8}$$

Analogously,

$$Y \circ X = \sum_{p,q=1}^{D} g^p \frac{\partial f^q}{\partial x^p} \frac{\partial}{\partial x^q} + \sum_{p,q=1}^{D} f^p g^q \frac{\partial^2}{\partial x^q \partial x^p}. \tag{10.9}$$

Because of the symmetry of second derivatives, the second order terms in $[X, Y]$ cancel. Hence,

$$[X, Y] = \sum_{p,q=1}^{D} (f^p \frac{\partial g^q}{\partial x^p} - g^p \frac{\partial f^q}{\partial x^p}) \frac{\partial}{\partial x^q}. \tag{10.10}$$

This can also be written as

$$[X, Y] = \sum_{q=1}^{D} (X g^q - Y f^q) \frac{\partial}{\partial x^q}. \tag{10.11}$$

Part III
Meshes

11 Meshes

Definition 11.1. Let d be a natural number. A **mesh** of dimension d is a finite set of polytopes of dimension at most d such that:

- if X is an element of M, then all subfaces of X are also in M;
- the intersection of elements of M is a finite (possibly empty) union of elements of M.

For an integer $p \in [0, d]$, the set of elements (polytopes) of dimension p in M is denoted by M_p.

12 Relative orientation on meshes

Theorem 12.1. Let $d \in \mathbb{N}$, M be a mesh of dimension d, $p \in \{2, ..., d\}$, $a_p \in M_p$, $c_{p-2} \in M_{p-2}$, $a_p \succ c_{p-2}$. Then there exist exactly two $(p-1)$-cells $b_{p-1}, b'_{p-1} \in M_{p-1}$ that are between a_p c_{p-2}, i.e.,

$$a_p \succ b_{p-1} \succ c_{p-2} \text{ and } a_p \succ b''_{p-1} \succ c_{p-2}. \tag{12.1}$$

Definition 12.2. Let $d \in \mathbb{N}$, M be a mesh of dimension d, R be a commutative ring with unity. A family of maps

$$\{\varepsilon_p \colon M_p \times M_{p-1} \to \{-1, 0, 1\}\}_{p=1}^{d} \tag{12.2}$$

is called a **relative orientation** on M if the following conditions are satisfied:

1. for any $p \in \{1, ..., d\}$, $a_p \in M_p$, $b_{p-1} \in M_{p-1}$,

$$a_p \succ b_{p-1} \Leftrightarrow \varepsilon(a_p, b_{p-1}) \neq 0; \tag{12.3}$$

2. for any edge a_1 with endpoints the nodes b_0 and c_0,

$$\varepsilon(a_1, b_0) + \varepsilon(a_1, c_0) = 0; \tag{12.4}$$

3. for any $p \in \{2, ..., d\}$, $a_p \in M_p$, $c_{p-2} \in M_{p-2}$ with $a_p \succ c_{p-2}$, let b_{p-1} and b'_{p-1} be the two cells between a_p and c_{p-2}. Then

$$\varepsilon(a_p, b_{p-1})\varepsilon(b_{p-1}, c_{p-2}) + \varepsilon(a_p, b'_{p-1})\varepsilon(b'_{p-1}, c_{p-2}) = 0. \tag{12.5}$$

Remark 12.3. Note that the last condition in the above definition can be written as

$$\sum_{b_{p-1} \in (c_{p-2}, a_p)} \varepsilon(a_p, b_{p-1})\varepsilon(b_{p-1}, c_{p-2}) = 0. \tag{12.6}$$

Theorem 12.4. Let $d \in \mathbb{N}$, M be a mesh of dimension d, R be a commutative ring with unity. Then there exists a relative orientation on M.

13 Chains and boundary operator on meshes

Definition 13.1. Let $d \in \mathbb{N}$, M be a mesh of dimension d, $p \in \mathbb{N}$, $p \in [0, d]$, R be a commutative ring with unity (for instance, $R = \mathbb{R}$). The space $C_p(M; R)$ of **p-chains** on M with coefficients in R is the free R-module (vector space over R when R is a field, e.g., when $R = \mathbb{R}$) generated by M_p:

$$C_p(M; R) := \mathrm{Free}_R(M_p). \tag{13.1}$$

In other words, the elements of $C_p(M; R)$ are the formal linear combinations of cells in M_p in coefficients in R. An element c_p of $C_p(M; R)$ has the form

$$c_p := \lambda_0 c(p, h_0) + ... + \lambda_{n-1} c(p, h_{n-1}), \tag{13.2}$$

where for $i = 0, ..., n-1$, $\lambda_i \in R$ and $c(p, h_i) \in M_p$.

Definition 13.2. Let $d \in \mathbb{N}$, M be a mesh of dimension d, R be a commutative ring with unity. The space $C_\bullet(M; R)$ of all **chains** on M is the direct sum

$$C_\bullet(M; R) := \bigoplus_{p=0}^{d} C_p(M; R). \tag{13.3}$$

Definition 13.3. Let $d \in \mathbb{N}$, M be a mesh of dimension d, R be a commutative ring with unity, ε be a relative orientation on M, $p \in \{1, ..., d\}$, The **boundary operator on p-cells** ∂_p is the map

$$\partial_p \colon C_p(M; R) \to C_{p-1}(M; R) \tag{13.4}$$

defined for a basis cochain $c_p \in M_p$ by

$$\partial_p a_p := \sum_{b_{p-1} \preceq a_p} \varepsilon(a_p, b_{p-1}) b_{p-1}. \tag{13.5}$$

and extended on $C_p M$ by linearity.
The full **boundary operator** ∂ on M,

$$\partial \colon C_\bullet(M; R) \to C_\bullet(M; R), \tag{13.6}$$

is the sum of all boundary operators on p-cells. In other words, for any $p \in \{1, ..., d\}$ $\sigma_p \in C_p(M; R)$,

$$\partial \sigma_p := \partial_p \sigma_p, \tag{13.7}$$

and ∂ is extended by linearity on all cells (it returns zero when acting on 0-cells).

Proposition 13.4. Let $d \in \mathbb{N}$, M be a mesh of dimension d, R be a commutative ring with unity, ε be a relative orientation on M. Then the algebra $(C_\bullet M, \partial)$ is a chain complex, i.e.,

$$\partial^2 = 0. \tag{13.8}$$

Proof. It is enough to prove that for any $p \in \{0, ..., d\}$, $c_p \in M_d$,

$$\partial^2 c_p = 0. \tag{13.9}$$

The proposition is trivially true for $p = 0$ and $p = 1$ because $\partial_0 = 0$. Assume that $p \geq 2$. Then

$$\begin{aligned}
\partial^2 a_p &= \partial_{p-1}(\partial_p a_p) \\
&= \partial_{p-1}\left(\sum_{b_{p-1} \prec a_p} \varepsilon(a_p, b_{p-1}) b_{p-1}\right) \\
&= \sum_{b_{p-1} \prec a_p} \sum_{c_{p-2} \prec b_{p-1}} \varepsilon(a_p, b_{p-1}) \varepsilon(b_{p-1}, c_{p-2}) c_{p-2} \\
&= \sum_{c_{p-2} \prec a_p} \left(\sum_{b_{p-1} \in (c_{p-2}, a_p)} \varepsilon(a_p, b_{p-1}) \varepsilon(b_{p-1}, c_{p-2})\right) c_{p-2} \\
&= 0
\end{aligned} \tag{13.10}$$

(the last equation follows from the last condition in the definition of relative orientation). □

Proposition 13.5. Let $d \in \mathbb{N}$, M be a mesh of dimension d, R be a commutative ring with unity, ε and ε' be a relative orientations on M with corresponding boundary operators ∂ and ∂' respectively. Then

$$(C_\bullet(M; R), \partial) \cong (C_\bullet(M; R), \partial') \tag{13.11}$$

($\cong$ is understood as isomorphism of chain complexes).

Remark 13.6. The above proposition says that the boundary operator is essentially unique, i.e., up to isomorphism it does not depend on the chosen relative orientation. This motivates the notion of "*the* boundary operator" on a mesh. Nevertheless, this does not exclude special choices of relative orientations in some particular cases like compatibly orientable meshes or regular grids.

14 Cochains and coboundary operator on meshes

Definition 14.1. Let $d \in \mathbb{N}$, M be a mesh of dimensions d, R be a commutative ring with unity. The space $C^\bullet(M; R)$ of **cochains** is the dual to the space of chains, i.e.,

$$C^\bullet(M; R) := (C_\bullet(M; R))^* = \mathrm{Hom}_R(C_\bullet(M; R), R) \tag{14.1}$$

If $p \in \{0, .., d\}$ then the space of **p-cochains** $C^p(M; R)$ is

$$C^p(M; R) := (C_p(M; R))^* = \mathrm{Hom}_R(C_p(M; R), R). \tag{14.2}$$

We have the decomposition

$$C^\bullet(M; R) = \bigoplus_{p=0}^{d} C^p(M; R). \tag{14.3}$$

Definition 14.2. Let $d \in \mathbb{N}$, M be a mesh of dimension d, R be a commutative ring with unity, ∂ be a boundary operator on M. Then the corresponding **coboundary operator** on M δ is the dual of ∂, i.e.,

$$\delta = \partial^* : C^\bullet(M; R) \to C^\bullet(M; R). \tag{14.4}$$

In other words, for any cochain $\pi^\bullet \in C^\bullet(M; R)$ and any chain $\rho_\bullet \in C_\bullet(M; R)$,

$$(\delta \pi^\bullet) \rho_\bullet := \pi^\bullet(\partial \rho_\bullet). \tag{14.5}$$

If $p \in \{0, ..., d-1\}$, the **coboundary operator on p-cochains** δ_p is defined as the dual of ∂_{p+1}. In other words,

$$\delta_p = \partial_{p+1}^* : C^p(M; R) \to C^{p+1}(M; R). \tag{14.6}$$

If $\pi^p \in C^p(M; R)$, $\rho_{p+1} \in C_{p+1}(M; R)$, then

$$(\delta_p \pi^p) \rho_{p+1} = \pi^p(\partial_{p+1} \rho_{p+1}). \tag{14.7}$$

Proposition 14.3. Let $d \in \mathbb{N}$, M be a mesh of dimension d, R be a commutative ring with unity, ∂ be a boundary operator on M, δ be the corresponding coboundary operator. Then

$$(C^\bullet(M; R), \delta) \tag{14.8}$$

is a cochain complex.

Proof. Let $\pi^\bullet \in C^\bullet(M; R)$, $\rho_\bullet \in C_\bullet(M; R)$. Then

$$(\delta^2 \pi^\bullet)\rho_\bullet = (\delta\pi^\bullet)(\partial\rho_\bullet) = \pi^\bullet(\partial(\partial\rho_\bullet)) = \pi^\bullet(0) = 0. \tag{14.9}$$

Since $\pi^\bullet$ and $\rho_\bullet$ were arbitrary,

$$\delta^2 = 0. \tag{14.10}$$

□

15 Combinatorial differential forms and Forman subdivision

Definition 15.1. Let $D \in \mathbb{N}$, M be a mesh of dimension D, $0 \leq p_f \leq D$. A **combinatorial differential form** of dimension p_f (called p_f-form for short) on M is a linear map

$$\omega \colon C_\bullet M \to C_\bullet M \tag{15.1}$$

such that for any $p \in [p_f, D]$ and any $c_p \in M_p$, $\omega(c_p)$ is a linear combination of the $(p - p_f)$-subfaces of c_p.
The space of all p_f-forms on M is denoted by $\Omega^{p_f} M$. The space of all combinatorial differential forms is the direct sum

$$\Omega^\bullet M := \bigoplus_{p_f=0}^{D} \Omega^{p_f} M. \tag{15.2}$$

Definition 15.2. Let $d \in \mathbb{N}$, M be a mesh of dimension d ∂ be a boundary operator on M. The **discrete differential** on M is the linear map

$$D \colon \Omega^\bullet \to \Omega^\bullet \tag{15.3}$$

which maps a p-form ω to a $(p+1)$-form by the formula

$$D\omega := \omega \circ \partial - (-1)^p \partial \circ \omega. \tag{15.4}$$

Proposition 15.3. Let $d \in \mathbb{N}$, M be a mesh of dimension d, ∂ be a boundary operator on Mm D be the discrete differential on M. Then $(\Omega^\bullet M, D)$ is a cochain complex, i.e.,

$$D^2 = 0. \tag{15.5}$$

Proof. A straightforward computation using the fact that $\partial^2 = 0$. Indeed, let $\omega \in \Omega^p M$. Then

$$\begin{aligned} D^2(\omega) &= D(D\omega) \\ &= D(\omega \circ \partial - (-1)^p \partial \circ \omega) \\ &= \omega \circ \partial \circ \partial - (-1)^{p+1} \partial \circ \omega \circ \partial - (-1)^p \partial \circ \omega \circ \partial - (-1)^p (-1)^{p+1} \partial \circ \partial \circ \omega \\ &= 0 - (-1)^{p+1} \partial \circ \omega \circ \partial + (-1)^{p+1} \partial \circ \omega \circ \partial + 0 \\ &= 0. \end{aligned} \tag{15.6}$$

□

Definition 15.4. Let $d \in \mathbb{N}$, M be a mesh of dimension d. Consider a mesh K constructed as follows. The nodes of K are the centroids of the cells of M. (In general, the topology of K can always be constructed while the geometry is tricky. For simplicity we may assume that all the cells of M are convex, although for non-simplicial or non-brick meshes in dimensions 3 and above the resulting mesh may contain non-flat polytopes.)
For $p_f \in [0, d]$, a p_f-cell of K is constructed as follows. Let $p \in [p_f, d]$, $s = p - p_f$ consider two cells

$$c(p, i) \succeq c(s, l) \tag{15.7}$$

Then a p_f-cell is such a pair $(c(p,i), c(s,l))$. If $q_f \in [0, p_f]$, $q \in [q_f, d]$, $r = q - q_f$,

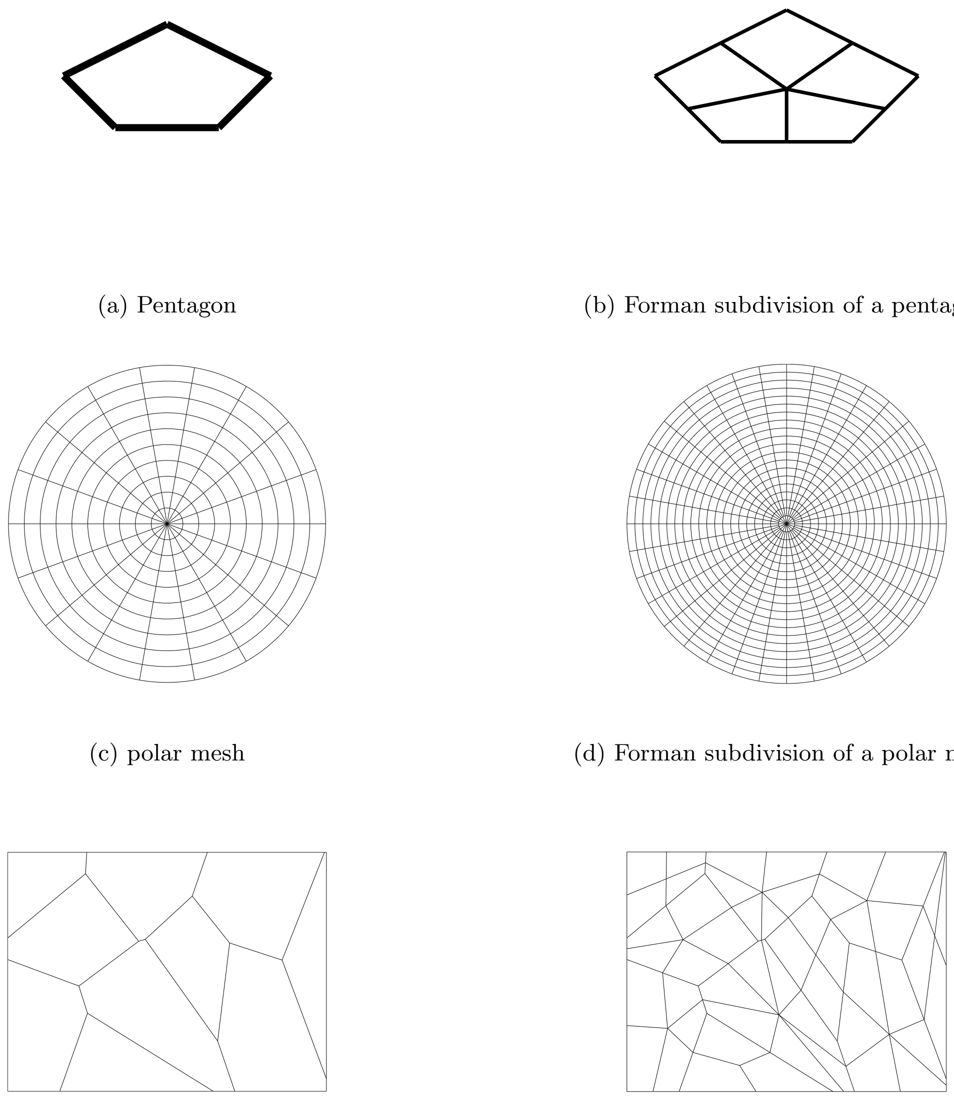
$$c(q,j) \succeq c(r,k), \tag{15.8}$$

then $(c(q,j), c(r,k))$ is a subface of $(c(p,i), c(s,l))$ if

$$c(p,i) \succeq c(q,j) \succeq c(r,k) \succeq c(s,l). \tag{15.9}$$

The constructed space is a mesh which we call the **Forman subdivision**.

Example 15.5. Some examples with planar meshes and their Forman subdivisions are presented in Figure 1.



(a) Pentagon

(b) Forman subdivision of a pentagon

(c) polar mesh

(d) Forman subdivision of a polar mesh

(e) Irregular mesh (produced by Neper)

(f) Forman subdivision of an irregular mesh

Figure 1: Examples of meshes and their Forman subdivisions

Definition 15.6. Let $d \in \mathbb{N}$, M be a mesh of dimension d, K be the Forman subdivision of M, ε_M be the relative orientation on M. We construct the relative orientation ε_K as follows. Let $p_f \in [1, d]$, $p \in [p_f, d]$, $s = p - p_f$, $c_K(p_f, i_f)$ be a p_f-cell on K,

$$c_K(p_f, i_f) = (c(p,i), c(s,l)) \text{ for some } c(p,i) \in C_p M \text{ and } c(s,l) \in C_s M. \tag{15.10}$$

Let $c_K(p_f - 1, j_f)$ be a hyperface of $c_K(p_f, i_f)$. Then there exist $q, r \in \mathbb{N}$ such that $p \geq q \geq r \geq s$ and $q - r = p_f - 1$, such that

$$c_K(p_f - 1, j_f) = (c(q, j), c(r, k)) \text{ for some } c(q, j) \in C_p M \text{ and } c(r, k) \in C_r M. \tag{15.11}$$

There are two possibilities for q and r: $(q, r) = (p - 1, s)$ or $(q, r) = (p, s + 1)$.

1. If $(q, r) = (p - 1, s)$, then

$$c_K(p_f - 1, j_f) = (c(p - 1, j), c(s, l)), \text{ where } c(p, i) \succ c(p - 1, j) \succeq c(s, l). \tag{15.12}$$

   In this case

$$\varepsilon_K(c_K(p_f, i_f), c_K(p_f - 1, j_f)) = \varepsilon_M(c_M(p, i), c_M(p - 1, j)). \tag{15.13}$$

2. If $(q, r) = (p, s + 1)$, then

$$c_K(p_f - 1, j_f) = (c(p, i), c(s + 1, k)), \text{ where } c(p, i) \succeq c(s + 1, k) \succ c(s, l). \tag{15.14}$$

   In this case

$$\varepsilon_K(c_K(p_f, i_f), c_K(p_f - 1, j_f)) = (-1)^{p_f} \varepsilon_M(c_M(s + 1, k), c_M(s, l)). \tag{15.15}$$

Theorem 15.7. Let $d \in \mathbb{N}$, M be a mesh of dimension d, ε_M be a relative orientation on M with corresponding boundary operator ∂_M and discrete differential D_M. Let K be the Forman subdivision of M, ε_K be the orientation on K constructed above, d_K be the corresponding coboundary operator on K. Then

$$(\Omega^p M, D_M) \cong (C^p K, d_K). \tag{15.16}$$

with the isomorphism being the mapping of the basis forms to basis cochains introduced in the construction of K.

Definition 15.8. Let $d \in \mathbb{N}$, P be a polytope of dimension d. We say that P is a (d-dimensional) **quasi-cube** if the mesh topology of P is isomorphic to the mesh topology of the d-cube.

Example 15.9. All 0D and 1D polytopes (points and segments respectively) are quasi-cubes. In 2D quasi-cubes are quadrilaterals. In 3D quasi-cubes are shapes with 8 vertices, 12 edges and 6 faces (hexahedra).

Definition 15.10. Let M be a mesh. We say that M is a **quasi-cubical mesh** if all cells of M are quasi-cubes.

Definition 15.11. Let $\mathcal{K}$ be a quasi-cubical mesh, $p \in \{1, ..., \dim \mathcal{K}\}$, $a \in \mathcal{K}_p$, $b, c \in \mathcal{K}_{p-1}$, $b, c \prec a$. We say that b and c are **topologically parallel** (with respect to a) if their intersection is empty. In that case we will write

$$a \setminus b := c, \ \ a \setminus c := b. \tag{15.17}$$

Definition 15.12. Let $(P, \preceq)$ be a partially ordered set, $a, b \in P$ with $a \preceq b$. The **closed interval** $[a, b]$ is defined as

$$[a, b] := \{x \in P \mid a \preceq x \ \& \ x \preceq b\}. \tag{15.18}$$

Definition 15.13. We say that a mesh M is **interval-simplicial** if for any two cells $a, b \in M$ with $a \preceq b$, the interval $[a, b]$ is an abstract simplex.

Proposition 15.14. Let M be a mesh and K be its Forman subdivision. Then M is interval-simplicial if and only if K is quasi-cubical.

Proposition 15.15. Let M be a mesh of dimension at most 2. Then M is interval-simplicial. In particular, its Forman subdivision is quasi-cubical.

Definition 15.16. Let $d \in \mathbb{N}$ and P be a polytope of dimension d. We say that P is a **simple polytope** if any of its nodes is connected to exactly d edges.

Proposition 15.17. Let M be a mesh of dimension 3. Then M is interval-simplicial if and only if all 3-cells of M are simple polytopes. In particular, if K is the Forman subdivision of M, then K is quasi-cubical if and only if all 3-cells of M are simple polytopes.

Proposition 15.18. Let $D \in \mathbb{N}$, K be a quasi-cubical mesh of dimension D, ∂ be the topological (unsigned) boundary operator on K, δ be the topological (unsigned) coboundary operator on K, $\perp$ be the perpendicularity operator on K, $p \in \{0, ..., D\}$. Then

$$\partial_{D-p} \circ \perp_p = \perp_{p+1} \circ \delta_p. \tag{15.19}$$

16 Metric-dependent calculus on quasi-cubical meshes

Discussion 16.1. As we saw, the Forman subdivision of an interval-simplicial mesh leads to a quasi-cubical mesh. Interval-simplicial meshes are not that uncommon:

1. all meshes of dimension at most 2 are interval-simplicial;
2. all 3D meshes of simple polytopes are interval-simplicial;
3. all simplicial and quasi-cubical meshes are interval-simplicial;
4. the product of interval-simplicial meshes is an interval-simplicial mesh.

For that reason we will build our calculus on quasi-cubical meshes thought as the Forman subdivision of an interval-simplicial mesh.

Definition 16.2. Let $D \in \mathbb{N}$, K be a quasi-cubical mesh of dimension D, $a_D \in M_D$, $p \in \{0, ..., D\}$, $b_p \in M_p$, $c_{D-p} \in M_{D-p}$. We say that b_p and c_{D-p} are **topologically orthogonal** with respect to a_D if $b_p, c_{D-p} \preceq a_D$, and the intersection of b_p with c_{D-p} is a node in a_D. In this case we write

$$b_p \oplus c_{D-p} = a_p \text{ and } b_p \perp c_{D-p}. \tag{16.1}$$

Notation 16.3. Let $D \in \mathbb{N}$, P be a polytope of dimension D. The **(Euclidean) measure** of P is denoted by

$$\mu_d(P)(= \mu(P)). \tag{16.2}$$

If P is with standard physical dimensions (e.g., it has not been non-dimensionalised), then $\mu_d(P)$ is of physical dimension $[L^D]$.

Definition 16.4. Let $D \in \mathbb{N}$, K be a quasi-cubical mesh of dimension D, $p \in \{0, ..., D\}$. The **inner product of p-cochains** $\langle \cdot, \cdot \rangle_p \colon C^p K \times C^p K \to C^p K$ is defined as follows. If $b_p \in K_p$, then

$$\langle b^p, b^p \rangle_p := \frac{1}{2^D \mu(b_p)} \sum_{c_{D-p} \perp b_p} \mu(c_{D-p}). \tag{16.3}$$

The inner product between two different p-cochains in the standard basis is defined to be zero (hence, the standard cochain basis is an orthogonal basis with respect to the inner product). Extending by bilinearity, we define the inner product of arbitrary p-cochains (since we already have it for pairs of basis p-cochains).
The operator $\langle \cdot, \cdot \rangle_p$ has physical dimension $[L^{D-2p}]$.

Example 16.5. Let $d \in \mathbb{N}$, $h \in \mathbb{R}^+$ K be a regular cubical mesh of dimension d with size h, $p \in \{0, ..., d\}$, b_p be an internal p-cell in K. Then

$$\langle b^p, b^p \rangle_p = h^{d-2p}. \tag{16.4}$$

Definition 16.6. Let $d \in \mathbb{N}$, K be a quasi-cubical mesh of dimension d, ∂ be a boundary operator on K, $p \in \{1, ..., d\}$. The **adjoint coboundary operator on p-cochain** $\delta_p^\star \colon C^p K \to C^{p-1} K$ is defined as the adjoint of δ_{p-1} with respect to the inner product of cochains. In other words, for any $\pi^p \in C^p K$, $\rho^{p-1} \in C^{p-1}$,

$$\langle \delta_p^\star \pi^p, \rho^{p-1} \rangle_{p-1} = \langle \pi^p, \delta_{p-1} \rho^{p-1} \rangle_p. \tag{16.5}$$

The operator $\delta_p^\star$ has physical dimension $[L^{-2}]$.

Proposition 16.7. Let $d \in \mathbb{N}$, K be a quasi-cubical mesh of dimension d, ε be a relative orientation on K, $p \in \{1, ..., d\}$, $\pi^p \in C^p K$, $b_{p-1} \in K_{p-1}$ Then

$$(\delta_p^\star \pi^p) b_{p-1} = \sum_{a_p \succ b_{p-1}} \frac{\langle a^p, a^p \rangle}{\langle b^{p-1}, b^{p-1} \rangle} \varepsilon(a_p, b_{p-1}) \pi^p(a_p). \tag{16.6}$$

Corollary 16.8. Let $d \in \mathbb{N}$, $h \in \mathbb{R}^+$ K a cubical grid of dimension d with size h with its standard orientation, $p \in \{0, ..., d\}$, $\pi^p \in C^p K$, $b_{p-1} \in K_{p-1}$ be an internal cell. Then

$$(\delta_p^\star \pi^p) b_{p-1} = \frac{1}{h^2} \sum_{b_{p-1} \prec a_p} \varepsilon(a_p, b_{p-1}) \pi^p(a_p). \tag{16.7}$$

Definition 16.9. Let $D \in \mathbb{N}$, K be a compatibly oriented quasi-cubical mesh of dimension D, $[K] := \sum_{c_D \in K_D} c^D$ be the fundamental class of K $\langle \cdot, \cdot \rangle$ be an inner product on K, $p \in \{0, ..., D\}$. The **Hodge star operator on p-cochains** $\star_p \colon C^p K \to C^{D-p} K$ is defined as the unique map satisfying the following equation: for any $\pi^p \in C^p K$ and $\rho^{D-p} \in C^{D-p} K$,

$$\langle \star_p \pi^p, \rho^{D-p} \rangle_{D-p} = (\pi^p \smile \rho^{D-p})[K]. \tag{16.8}$$

The operator $\star_p$ has physical dimension $[L^{D-2p}]$.

Proposition 16.10. Let $D \in \mathbb{N}$, K be a compatibly oriented quasi-cubical mesh of dimension D, $[K] := \sum_{c_D \in K_D} c^D$ be the fundamental class of K $\langle \cdot, \cdot \rangle$ be an orthogonal inner product on K, $p \in \{0, ..., D\}$. The Hodge star operator $\star_p \colon C^p K \to C^{D-p} K$ has the following closed form: for any $\pi^p \in C^p K$ and any $c^{D-p} \in C^{D-p} K$,

$$(\star_p \pi^p)(b_{D-p}) = \sum_{a_p \perp b_{D-p}} \frac{(a^p \smile b^{D-p})[K]}{\langle b^{D-p}, b^{D-p} \rangle} \pi^p(a_p). \tag{16.9}$$

17 Product meshes

Definition 17.1. Let $\mathcal{K}$ and $\mathcal{L}$ be meshes. The **product mesh** $\mathcal{K} \times \mathcal{L}$ is defined as the poset product of $\mathcal{K}$ and $\mathcal{L}$.

Proposition 17.2. Let $\mathcal{K}$ and $\mathcal{L}$ be meshes. Then $\mathcal{K} \times \mathcal{L}$ is also a mesh.

Proposition 17.3. Let $\mathcal{K}$ and $\mathcal{L}$ be quasi-cubical meshes. Then $\mathcal{K} \times \mathcal{L}$ is also a quasi-cubical mesh.

Proposition 17.4. Let $\mathcal{M}$ and $\mathcal{N}$ be combinatorial meshes. Then

$$\mathrm{Forman}(\mathcal{M} \times \mathcal{N}) = \mathrm{Forman}(\mathcal{M}) \times \mathrm{Forman}(\mathcal{N}). \tag{17.1}$$

Definition 17.5. Let $\varphi \colon \mathcal{K} \to \mathcal{P}K$ and $\psi \colon \mathcal{L} \to \mathcal{P}L$ be mesh embeddings. Define the **product embedding**

$$\varphi \times \psi \colon \mathcal{K} \times \mathcal{L} \to \mathcal{P}(K \times L) \tag{17.2}$$

as follows: for any $(a, \alpha) \in \mathcal{K} \times \mathcal{L}$,

$$(\varphi \times \psi)(a, \alpha) = (\varphi a) \times (\psi \alpha). \tag{17.3}$$

Proposition 17.6. Let $(\mathcal{K}, \varepsilon^{\mathcal{K}})$ and $(\mathcal{K}, \varepsilon^{\mathcal{K}})$ be meshes with relative orientations, $C_\bullet(\mathcal{K}, \partial^{\mathcal{K}})$ and $C_\bullet(\mathcal{L}, \partial^{\mathcal{L}})$ be the corresponding chain complexes (of boundary operators). Then the chain complex

$$C_\bullet(\mathcal{K}, \partial^{\mathcal{K}}) \otimes C_\bullet(\mathcal{L}, \partial^{\mathcal{L}}) \tag{17.4}$$

induces a relative orientations on $\mathcal{K} \times \mathcal{L}$. Precisely, if $0, \leq p \leq \dim \mathcal{K}$, $0, \leq q \leq \dim \mathcal{L}$, $a \in \mathcal{K}_p, \alpha \in \mathcal{L}_q$, $(b, \beta) \in \partial_{p+q}(a, \alpha)$,

$$\varepsilon_{p+q}^{\mathcal{K} \times \mathcal{L}}((a, \alpha), (b, \beta)) = \begin{cases} \varepsilon_p^{\mathcal{K}}(a, b), & \alpha = \beta \\ (-1)^p \varepsilon_q^{\mathcal{L}}(\alpha, \beta), & a = b \end{cases}. \tag{17.5}$$

Definition 17.7. Let $\mathcal{K}, \mu^{\mathcal{K}}$ and $\mathcal{L}, \mu^{\mathcal{L}}$ be Riemannian meshes. Define the **product measures** $\mu^{\mathcal{K}} \times \mu^{\mathcal{L}}$ on $\mathcal{K} \times \mathcal{L}$ as follows: for any $a \in \mathcal{K},\ \alpha \in \mathcal{L}$,

$$(\mu^{\mathcal{K}} \times \mu^{\mathcal{L}})(a, \alpha) := \mu^{\mathcal{K}}(a)\, \mu^{\mathcal{L}}(\alpha). \tag{17.6}$$

Proposition 17.8. Let $\mathcal{K}$ and $\mathcal{L}$ be combinatorial meshes, (K, g^K) and (L, g^L) be Riemannian manifolds, $\varphi^{\mathcal{K}} \colon \mathcal{K} \to K$ and $\varphi^{\mathcal{L}} \colon \mathcal{L} \to L$ be mesh embeddings, $\mu^{\mathcal{K}}$ and $\mu^{\mathcal{L}}$ be the induced measures. Then $\mu^{\mathcal{K}} \times \mu^{\mathcal{L}}$ is the measure induced by the embedding $\varphi^{\mathcal{K}} \times \varphi^{\mathcal{L}}$ of the mesh $\mathcal{K} \times \mathcal{L}$ in the Riemannian manifold $(K \times L, g^K \times g^L)$.

Proposition 17.9. Let $(\mathcal{K}, \mu^{\mathcal{K}})$ and $(\mathcal{L}, \mu^{\mathcal{L}})$ be Riemannian meshes, $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ be the respective induced inner products, $\langle \cdot, \cdot \rangle_{\mathcal{K} \times \mathcal{L}}$ be the inner product induced by $\mu^{\mathcal{L}} \times \mu^{\mathcal{L}}$, $(a, \alpha) \in \mathcal{K} \times \mathcal{L}$. Then

$$\langle (a, \alpha)^\bullet, (a, \alpha)^\bullet \rangle_{\mathcal{K} \times \mathcal{L}} = \langle a^\bullet, a^\bullet \rangle_{\mathcal{K}} \, \langle \alpha^\bullet, \alpha^\bullet \rangle_{\mathcal{L}}. \tag{17.7}$$

Proof. Let $D = \dim \mathcal{K}$, $d = \dim \mathcal{L}$. Then $\dim(\mathcal{K} \times \mathcal{L}) = D + d$. Hence,

$$\begin{aligned}
\langle (a, \alpha)^\bullet, (a, \alpha)^\bullet \rangle_{\mathcal{K} \times \mathcal{L}} &= \frac{1}{2^{D+d} \mu^{\mathcal{K} \times \mathcal{L}}(a, \alpha)} \sum_{(b, \beta) \perp (a, \alpha)} \mu^{\mathcal{K} \times \mathcal{L}}(b, \beta) \\
&= \frac{1}{2^D \mu^{\mathcal{K}}(a)} \frac{1}{2^d \mu^{\mathcal{L}}(\alpha)} \sum_{b \perp a,\ \beta \perp \alpha} \mu^{\mathcal{K}}(b)\, \mu^{\mathcal{L}}(\beta) \\
&= \left(\frac{1}{2^D \mu^{\mathcal{K}}(a)} \sum_{b \perp a} \mu^{\mathcal{K}}(b) \right) \left(\frac{1}{2^d \mu^{\mathcal{L}}(\alpha)} \sum_{\beta \perp \alpha} \mu^{\mathcal{L}}(\beta) \right) \\
&= \langle a^\bullet, a^\bullet \rangle_{\mathcal{K}} \, \langle \alpha^\bullet, \alpha^\bullet \rangle_{\mathcal{L}}.
\end{aligned} \tag{17.8}$$

□

Proposition 17.10. Let $\mathcal{K}$ and $\mathcal{L}$ be combinatorial meshes, (K, g^K) and (L, g^L) be smooth Riemannian manifolds that realise them, $\langle \cdot, \cdot \rangle_K$ and $\langle \cdot, \cdot \rangle_L$ be the induced inner products of differential forms, $W^{\mathcal{K}} \colon C^\bullet \mathcal{K} \to H\Omega^\bullet K$ and $W^{\mathcal{L}} \colon C^\bullet \mathcal{L} \to H\Omega^\bullet L$ be Whitney maps, $\langle \cdot, \cdot \rangle_{\mathcal{K}}$, $\langle \cdot, \cdot \rangle_{\mathcal{L}}$, and $\langle \cdot, \cdot \rangle_{\mathcal{K} \times \mathcal{L}}$ be the induced Whitney inner products (the last one is induced by $W^{\mathcal{K}} \otimes W^{\mathcal{L}}$), $\sigma, \tau \in C^\bullet \mathcal{K}$, $\omega, \eta \in C^\bullet \mathcal{L}$. Then

$$\langle \sigma \otimes \omega, \tau \otimes \eta \rangle_{\mathcal{K} \times \mathcal{L}} = \langle \sigma, \tau \rangle_{\mathcal{K}} \, \langle \omega, \eta \rangle_{\mathcal{L}}. \tag{17.9}$$

Proof. A direct computation:

$$\begin{aligned}
\langle \sigma \otimes \omega, \tau \otimes \eta \rangle_{\mathcal{K} \times \mathcal{L}} &= \langle W^{\mathcal{K} \times \mathcal{L}}(\sigma \otimes \omega), W^{\mathcal{K} \times \mathcal{L}}(\tau \otimes \eta) \rangle_{K \times L} \\
&= \langle W^{\mathcal{K}} \sigma \boxtimes W^{\mathcal{L}} \omega, W^{\mathcal{K}} \tau \boxtimes W^{\mathcal{L}} \eta \rangle_{K \times L} \\
&= \int_{K \times L} g^{K \times L}(W^{\mathcal{K}} \sigma \boxtimes W^{\mathcal{L}} \omega, W^{\mathcal{K}} \tau \boxtimes W^{\mathcal{L}} \eta)\ \mathrm{vol}_{K \times L} \\
&= \left(\int_K g^K(W^{\mathcal{K}} \sigma, W^{\mathcal{K}} \tau)\, \mathrm{vol}_K \right) \left(\int_L g^L(W^{\mathcal{L}} \omega, W^{\mathcal{L}} \eta)\, \mathrm{vol}_L \right) \\
&= \langle W^{\mathcal{K}} \sigma, W^{\mathcal{K}} \tau \rangle_K \, \langle W^{\mathcal{L}} \omega, W^{\mathcal{L}} \eta \rangle_L \\
&= \langle \sigma, \tau \rangle_{\mathcal{K}} \, \langle \omega, \eta \rangle_{\mathcal{L}}.
\end{aligned} \tag{17.10}$$

□

18 Approximating vector fields with 1-cochains

Definition 18.1. Let $m, n \in \mathbb{N}$, A be a real $m \times n$ matrix. An $n \times m$ matrix B is called **Moore-Penrose inverse** or **pseudo-inverse** if

$$ABA = A, \tag{18.1a}$$

$$BAB = B, \tag{18.1b}$$

$$(AB)^T = AB, \tag{18.1c}$$

$$(BA)^T = BA. \tag{18.1d}$$

Remark 18.2. Let A be a matrix that has a physical dimension [X], B be a Moore-Penrose inverse of A. Then B $[X^{-1}]$.

Theorem 18.3. Let $m, n \in \mathbb{N}$, A be a real $m \times n$ matrix. Then A has a unique Moore-Penrose inverse, denoted by A^*.

Remark 18.4. If the matrix A is of full rank there exists a closed formula for A^*.

1. If A is a square matrix with full rank, i.e., an invertible one, then $A^* = A^{-1}$.
2. If $m > n$ and A is an $m \times n$ matrix of full rank, then its columns are linearly independent which means that $A^T A$ is symmetric and positive definite and hence invertible. (Its inverse $(A^T A)^{-1}$ is also symmetric and positive definite.) It is then easy to check that

$$B := (A^T A)^{-1} A^T \tag{18.2}$$

is the Moore-Penrose inverse of A. Indeed, obviously B is a left inverse of A, and

$$\begin{aligned} ABA &= A(BA) = AI_n = A, &(18.3a)\\ BAB &= (BA)B = I_n B = B, &(18.3b)\\ (AB)^T &= (A(A^T A)^{-1} A^T)^T = (A^T)^T ((A^T A)^{-1})^T A^T = A(A^T A)^{-1} A^T = AB, &(18.3c)\\ (BA)^T &= I_n^T = I_n = BA. &(18.3d) \end{aligned}$$

3. If $m < n$ and A is an $m \times n$ matrix of full rank, then an analogous reasoning to the previous point shows that

$$A^* = A^T (AA^T)^{-1}. \tag{18.4}$$

Definition 18.5. Let $d \in \mathbb{N}$ and M be a mesh of dimension d. We say that M is **flat** if the following conditions are satisfied:

1. it is pure, i.e., all of its cells lie within some d-cell;
2. all its cells are flat (but possibly degenerate) polytopes;
3. it can be embedded in $\mathbb{R}^d$.

Definition 18.6. Let:

1. $d \in \mathbb{N},\ d \geq 1$;
2. K be a flat mesh of dimension d (with a chosen embedding in $\mathbb{R}^d$);
3. ϵ be a relative orientation on K;
4. $\mathcal{N}_i$ be a node in K connected to $n > 0$ edges;
5. $\mathcal{E}_{j_0}, ..., \mathcal{E}_{j_{n-1}}$ be all the edges containing $\mathcal{N}_i$ as a node;
6. $\mathcal{N}_{i_0}, ..., \mathcal{N}_{i_{n-1}}$ be the other than other $\mathcal{N}_i$ of $\mathcal{E}_{j_0}, ..., \mathcal{E}_{j_{n-1}}$ respectively.

We define the **node matrix** $\mathcal{L}_{\mathcal{N}_i}$ of $\mathcal{N}_i$ by

$$\mathcal{L}_{\mathcal{N}_i} := \begin{pmatrix} x_{i_0,0} - x_{i,0} & \cdots & x_{i_0,d-1} - x_{i,d-1} \\ \vdots & \ddots & \vdots \\ x_{i_{n-1},0} - x_{i,0} & \cdots & x_{i_{n-1},d-1} - x_{i,d-1} \end{pmatrix} \in \mathbb{R}^{n \times d}. \tag{18.5}$$

The physical dimension of $\mathcal{L}_{\mathcal{N}_i}$ is $[L]$.

Definition 18.7. Let $d \in \mathbb{N}$ with $d \geq 1$, K be a mesh of dimension d, $\pi^1 \in C^1 K$, $\mathcal{N}_i \in K_0$. Define the **neighbor representation** $\widehat{\pi^1}_{\mathcal{N}_i}$ of π^1 at $\mathcal{N}_i$ by

$$\widehat{\pi^1}_{\mathcal{N}_i} := (\epsilon_K(\mathcal{E}_{j_0}, \mathcal{N}_{i_0}) \pi^1(\mathcal{E}_{j_0}), \cdots, \epsilon_K(\mathcal{E}_{j_{k-1}}, \mathcal{N}_{i_{k-1}}) \pi^1(\mathcal{E}_{j_{k-1}})) \in \mathbb{R}^n. \tag{18.6}$$

The neighbor is a dimensionless operator.

Definition 18.8. Let $d \in \mathbb{N}$ with $d \geq 1$, K be a flat mesh of dimension d, $\pi^1 \in C^1K$, $\mathcal{N}_i \in K_0$ with corresponding Euclidean coordinates x_i. Define the **1-cochain embedding** $\overline{\pi^1}(x_i)$ of π^1 at $\mathcal{N}_i$ by

$$\overline{\pi^1}(x_i) := (\mathcal{L}_{\mathcal{N}_i})^* \, \widehat{\pi^1}_{\mathcal{N}_i} \in \mathbb{R}^d. \tag{18.7}$$

The 1-cochain embedding operator has physical dimension $[L^{-1}]$.

Example 18.9. Let $h \in \mathbb{R}^+$ and K be a regular subdivision with size h of some interval, all the edges in K are oriented from left to right, $\pi^1 \in C^1K$.

1. Consider an interior point $\mathcal{N}_i$ with neighboring edges $\mathcal{E}_{i-1}$ and $\mathcal{E}_i$ and corresponding neighboring nodes $\mathcal{N}_{i-1}$ and $\mathcal{N}_{i+1}$. Then

$$\mathcal{L}_{\mathcal{N}_i} = \begin{pmatrix} -h \\ h \end{pmatrix} \Rightarrow (\mathcal{L}_{\mathcal{N}_i})^* = \frac{1}{2h} \begin{pmatrix} -1 & 1 \end{pmatrix} \tag{18.8}$$

and

$$\widehat{\pi^1}_{\mathcal{N}_i} = \begin{pmatrix} -\pi^1\mathcal{E}_{i-1} \\ \pi^1\mathcal{E}_i \end{pmatrix} \Rightarrow \overline{\pi^1}(x_i) = \frac{1}{2h} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} -\pi^1\mathcal{E}_{i-1} \\ \pi^1\mathcal{E}_i \end{pmatrix} = \frac{1}{2h} \left(\pi^1\mathcal{E}_{i-1} + \pi^1\mathcal{E}_i\right). \tag{18.9}$$

Definition 18.10. Let K be an embedded flat mesh, X be the manifold enclosed by K, u be a vector field on X. Define the **approximation** $\underline{u} \in C^1K$ as follows. Let $\mathcal{E}_i$ be an edge with endpoints $\mathcal{N}_{i_0}$ and $\mathcal{N}_{i_1}$, oriented from $\mathcal{N}_{i_0}$ to $\mathcal{N}_{i_1}$. Let p_0 and p_1 be such that for $j = 0, 1$, the p_j-th edge adjacent to N_{i_j} is E_i. Then

$$\underline{u}(\mathcal{E}_i) := \frac{1}{2}(-(\mathcal{L}_{\mathcal{N}_{i_0}} u(x_{i_0}))_{p_0} + (\mathcal{L}_{\mathcal{N}_{i_1}} u(x_{i_1}))_{p_1}). \tag{18.10}$$

The approximation operator is of physical dimension $[L]$.

Example 18.11. With the mesh of the previous example, we have

$$\underline{u}(\mathcal{E}_i) = \frac{1}{2}(hu(x_i) + hu(x_{i+1})) = h\frac{u(x_i) + u(x_{i+1})}{2}. \tag{18.11}$$

Let's calculate the consecutive application of approximation and embedding (and vice versa).

$$\underline{\left(\overline{\pi^1}\right)}(\mathcal{E}_i) = h\left(\frac{\overline{\pi^1}(x_i) + \overline{\pi^1}(x_{i+1})}{2}\right) = \frac{h}{2}\frac{1}{2h}((\pi^1\mathcal{E}_{i-1}+\pi^1\mathcal{E}_i)+(\pi^1\mathcal{E}_i+\pi^1\mathcal{E}_{i+1})) = \frac{\pi^1\mathcal{E}_{i-1} + 2\pi^1\mathcal{E}_i + \pi^1\mathcal{E}_{i+1}}{4}. \tag{18.12}$$

$$\overline{(\underline{u})}(x_i) = \frac{1}{2h}\left(\underline{u}\mathcal{E}_{i-1} + \underline{u}\mathcal{E}_i\right) = \frac{1}{2h}\frac{h}{2}((u(x_{i-1})+u(x_i))+(u(x_i)+u(x_{i+1}))) = \frac{u(x_{i-1}) + 2u(x_i) + u(x_{i+1})}{4}. \tag{18.13}$$

In both cases of composition of embedding and approximation the final result is the identity operator when π^1 (respectively u) is linear with respect to the index i.

Discussion 18.12. Let me summarize the operations relating cochains and embedding. We will use notation coming from dependent type theory for functions whose codomain depends on the domain. Namely, if X is a type (set), $\{Y(x)\}_{x \in X}$ is a family of sets and $\{f(x) \in Y(x)\}_{x \in X}$, we will write

$$f\colon \prod_{x \in X} Y(x). \tag{18.14}$$

Let $d \in \mathbb{N}$, K be a quasi-cubical flat mesh of dimension d, X be the manifold it encompasses. We define the following data:

- $n\colon K_0 \to \mathbb{N}$ denotes the number of node neighbors of a 0-cell;
- $\widehat{\ }\colon C^1K \to \prod_{x_0 \in K_0} \mathbb{R}^{n(x_0)}$ denotes the neighbor representation of a 1-cochain, $\widehat{\ }$ [1];
- $\mathcal{L}\colon \prod_{x_0 \in K_0} M_{n(x_0),d}(\mathbb{R})$ denotes the node neighbors matrix, $\mathcal{L}$ $[L]$;

- $\star\colon \prod_{(m,n)\in\mathbb{N}^2} M_{m,n}(\mathbb{R}) \to M_{n,m}(\mathbb{R})$ denotes the Moore-Penrose inverse of a rectangular matrix, ($\star$ reverses physical dimensions);
- $\overline{}\colon C^1K \to \mathrm{Hom}_{\mathbb{R}}(C^0K, \mathbb{R}^d)$ denotes the approximation of a 1-cochain as a Euclidean vector-valued 0-cochain,

$$\overline{\pi^1}c_0 := (\mathcal{L}_{c_0})^\star \cdot (\widehat{\pi^1})_{c_0}, \tag{18.15}$$

$\overline{}$ $[L^{-1}]$;

- $\_\colon \chi X \to C^1K$ denotes the discretization of a continuum vector field as a 1-cochain, $\_$ $[L]$;

19 Vector fields on combinatorial meshes

Definition 19.1. Let $\mathcal{K}$ be a quasi-cubical mesh. We say that $\mathcal{X} \in \mathrm{Hom}(C_0\mathcal{K}, C_1\mathcal{K})$ is a **combinatorial vector field** if for any $\mathcal{N} \in \mathcal{K}_0$ and $\mathcal{E} \in \mathcal{K}_1$,

$$\mathcal{N} \notin \partial\mathcal{E} \Rightarrow \mathcal{E}^\bullet(\mathcal{X}\mathcal{N}_\bullet) = 0 \tag{19.1}$$

(here $a_\bullet$ and $a^\bullet$ denote the corresponding basis chains and cochains to a cell a). In other words, a combinatorial vector field assigns a basis 0-chain $\mathcal{N}_\bullet$ a 1-chain whose coefficients are zero on the edges that do not contain $\mathcal{N}$. We will write its coefficients by

$$\mathcal{X}^{\mathcal{E}}_{\mathcal{N}} := \mathcal{E}^\bullet(X\mathcal{N}_\bullet). \tag{19.2}$$

The space of all combinatorial vector fields will be denoted by $\mathfrak{X}K$.

Definition 19.2. Let $\mathcal{K}$ be a quasi-cubical mesh. Define the product of a function with a vector field as follows: for any $f \in C^0\mathcal{K}$, $\mathcal{X} \in \mathfrak{X}K$, $f\mathcal{X} \in \mathfrak{X}K$, and when applied at the basis chain of $\mathcal{N} \in \mathcal{K}_0$,

$$(f\mathcal{X})\mathcal{N}_\bullet := f\mathcal{N}\,\mathcal{X}\mathcal{N}_\bullet. \tag{19.3}$$

Proposition 19.3. Let $\mathcal{K}$ be a quasi-cubical mesh. Then the space $\mathfrak{X}K$ is a module over $C^0\mathcal{K}$.

Definition 19.4. Let $\mathcal{K}$ be a quasi-cubical mesh. The **discrete interior product on 1-cochains** is defined as

$$i\colon \mathfrak{X}K \to \mathrm{Hom}(C^1\mathcal{K}, C^0\mathcal{K}), i_{\mathcal{X}}\sigma := \sigma \circ \mathcal{X} \in C^0\mathcal{K},\ \ \mathcal{X} \in \mathfrak{X}K,\ \ \sigma \in C^1\mathcal{K}. \tag{19.4}$$

In other words, for a node $\mathcal{N}$,

$$(i_{\mathcal{X}}\sigma)(\mathcal{N}) = (\sigma \circ \mathcal{X})(\mathcal{N}) = \sum_{\mathcal{E}\succ\mathcal{N}} \mathcal{X}^{\mathcal{E}}_{\mathcal{N}}\sigma_{\mathcal{E}}. \tag{19.5}$$

Remark 19.5.

$$i_{f\mathcal{X}}\sigma = f \smile i_{\mathcal{X}}\sigma. \tag{19.6}$$

Indeed, for any $\mathcal{N} \in \mathcal{K}_0$,

$$(i_{f\mathcal{X}}\sigma)\mathcal{N}_\bullet = (\sigma \circ f\mathcal{X})\mathcal{N}_\bullet = \sigma(f\mathcal{N}_\bullet\,\mathcal{X}\mathcal{N}_\bullet) = f\mathcal{N}_\bullet\,\sigma(\mathcal{X}\mathcal{N}_\bullet) = (f \smile i_{\mathcal{X}}\sigma)\mathcal{N}_\bullet. \tag{19.7}$$

Definition 19.6. Let $D \in \mathbb{N}^+$, K be an orthogonal parallelotope in $\mathbb{R}^D$, whose unit directions are the vectors $e_1, ..., e_D$. Define a regular mesh for K as follows. Let $h_1, ..., h_D \in \mathbb{R}^+$, $\mathcal{K}$ be a grid of (orthogonal) parallelotopes with sides $h_1, ..., h_D$. Let $X \in \mathfrak{X}K$, $X = \sum_{p=1}^{D} f^p \frac{\partial}{\partial x^p}$. Define the approximation

$$J\colon \mathfrak{X}K \to \mathfrak{X}\mathcal{K} \tag{19.8}$$

as follows. Let $p \in \{1, ..., D\}$, $\mathcal{E}$ be an edge in the direction of the basis vector e_p, $\mathcal{N}$ be a node of $\mathcal{E}$ with coordinates

$x = (x_1, ..., x_D)$. Then,

$$(JX)^{\mathcal{E}}_{\mathcal{N}} := \begin{cases} \frac{f^p(x)}{2h_p}, & \mathcal{N} \text{ is an interior node} \\ \frac{f^p(x)}{h_p}, & \mathcal{N} \text{ is a boundary node} \end{cases}. \tag{19.9}$$

Remark 19.7. Consider the setup of the previous definition. We will show that the discrete interior product is a good approximation of the continuous one. Precisely, let $X \in \mathfrak{X}K$, $\omega \in \Omega^1 K$. We will compare $i_{JX} R_1 \omega$ with $R i_X \omega$. By definition, $i_X \omega = \omega(X)$. In coordinates,

$$X = \sum_{p=1}^{D} f^p \frac{\partial}{\partial x^p},\ f^1, ..., f^D \in \mathcal{F}K, \tag{19.10}$$

$$\omega = \sum_{p=1}^{D} g_p \, dx^p,\ g_1, ..., g_D \in \mathcal{F}K, \tag{19.11}$$

$$i_X \omega = \sum_{p=1}^{D} f^p \; g_p. \tag{19.12}$$

Consider a node $\mathcal{N}$ with coordinates $x = (x_1, ..., x_D)$. Then

$$(R i_X \omega)\mathcal{N} = \sum_{p=1}^{D} f^p(x_p) \; g_p(x_p). \tag{19.13}$$

On the other hand,

$$(i_{JX} R_1 \omega)\mathcal{N} = \sum_{\mathcal{E} \succ \mathcal{N}} (JX)^{\mathcal{E}}_{\mathcal{N}} \int_{\mathcal{E}} \omega = \sum_{p=1}^{D} \sum_{\mathcal{E} \succ \mathcal{N},\ \mathcal{E} \| e_p} (JX)^{\mathcal{E}}_{\mathcal{N}} \int_{\mathcal{E}} \omega. \tag{19.14}$$

In the above equation, for any $p \in \{1, ..., D\}$ the internal sum consists of 1 or 2 terms: 1 when $\mathcal{N}$ is on the boundary of the direction of e_p, and 2 elsewhere. Define the this internal sum as A_p. We need to show it is close to $f^p(x_p) \; g_p(x_p)$.
First, assume that $\mathcal{N}$ is in the interior. Then it is the boundary of two edges parallel to e_p: $\mathcal{E}_1$ and $\mathcal{E}_2$. Combined, they give the segment $\mathcal{E}$ connecting $x - h_p e_p$ with $x + h_p e_p$. Then

$$A_p = \frac{f^p(x)}{2h_p} \int_{\mathcal{E}} \omega = f^p(x_p) \frac{1}{2h_p} \int_{x_p - h_p}^{x_p + h_p} g_p(x_1, ..., x_{p-1}, t, x_{p+1}, ..., x_D) \, dt. \tag{19.15}$$

However, from 1D numerical analysis it follows that the approximation (the lowest order Gauss quadrature)

$$\frac{1}{b-a} \int_a^b g(t) \, dt \approx g\left(\frac{a+b}{2}\right) \tag{19.16}$$

is exact for polynomials of degree ≤ 1. In our case, substituting $a = x_p - h_p,\ b = x_p + h_p$ implies that $f^p(x_p) \; g_p(x_p)$ is an $O(h_p^2)$ approximation of A_p.
Second, assume that $\mathcal{E}$ is a boundary node. Fix it to be on the left. Then, using an analogous argument, we need to compare $g(a)$ with $\frac{1}{b-a} \int_a^b g(t) \, dt$ which is the formula of left rectangles, this time exact only for constants. This gives total error of $O(h_p)$ in the direction of e_p with $O(h^2)$ in the interior. Hence, the approximation formula is of order $O(h_1 + ... + h_D)$ with order $O(h_1^2 + ... + h_D^2)$ in the interior.

Definition 19.8. Let $\mathcal{K}$ be a quasi-cubical mesh. The **Lie derivative on 0-cochains** is defined as

$$L \colon \mathfrak{X}K \to \mathrm{Hom}(C^0\mathcal{K}, C^0\mathcal{K}), L_{\mathcal{X}} := i_{\mathcal{X}} \circ \delta,\ \mathcal{X} \in \mathfrak{X}K. \tag{19.17}$$

The above expression applied to a function $f \in C^0\mathcal{K}$ equals to

$$L_{\mathcal{X}} f = i_{\mathcal{X}}(\delta f) = (\delta_0 f) \circ \mathcal{X} = f \circ \partial_1 \circ X. \tag{19.18}$$

Remark 19.9. No we are going to analyse to what extent the Leibniz rule is reproduced in the discrete setting. Namely, for $\mathcal{X} \in \mathfrak{X}K,\ f, g \in \mathcal{F}K$ we are estimating

$$\Delta_{\mathcal{X}}(f, g) := (L_{\mathcal{X}} f) \smile g + f \smile (L_{\mathcal{X}} g) - L_{\mathcal{X}}(f \smile g). \tag{19.19}$$

Let $\mathcal{N} \in \mathcal{K}_0$. Then

$$\begin{aligned}\Delta_{\mathcal{X}}(f,g)\mathcal{N}_\bullet &= ((L_{\mathcal{X}}f) \smile g + f \smile (L_{\mathcal{X}}g) - L_{\mathcal{X}}(f \smile g))\mathcal{N}_\bullet \\ &= f(\partial(X\mathcal{N}_\bullet))\, g\mathcal{N}_\bullet + f\mathcal{N}_\bullet\, g(\partial(X\mathcal{N}_\bullet)) - (f \smile g)(\partial(X\mathcal{N}_\bullet)) \\ &= \sum_{\mathcal{E}\succ\mathcal{N}} \mathcal{X}^{\mathcal{E}}_{\mathcal{N}}((\sum_{\mathcal{N}'\prec\mathcal{E}} \varepsilon(\mathcal{E},\mathcal{N}')f_{\mathcal{N}'})g_{\mathcal{N}} + f_{\mathcal{N}}(\sum_{\mathcal{N}'\prec\mathcal{E}} \varepsilon(\mathcal{E},\mathcal{N}')g_{\mathcal{N}'}) - \sum_{\mathcal{N}'\prec\mathcal{E}} \varepsilon(\mathcal{E},\mathcal{N}')f_{\mathcal{N}'}g_{\mathcal{N}'}) \\ &= \sum_{\mathcal{E}\succ\mathcal{N}} \mathcal{X}^{\mathcal{E}}_{\mathcal{N}}\, \varepsilon(\mathcal{E},\mathcal{N})\ (\delta_0 f)(\mathcal{E})\,(\delta_0 g)(\mathcal{E}).\end{aligned} \tag{19.20}$$

Definition 19.10. Let $\mathcal{K}$ be a quasi-cubical mesh, ε be relative orientations on $\mathcal{K}$, $X, Y \in \mathcal{X}K$. Define the commutator $[X,Y] \in \mathcal{X}K$ as follows: for any edge $\mathcal{E}$ and node $\mathcal{N} \prec \mathcal{E}$,

$$[X,Y]^{\mathcal{E}}_{\mathcal{N}} := 2\varepsilon(\mathcal{E},\mathcal{N})(X^{\mathcal{E}}_{\mathcal{E}\setminus\mathcal{N}}Y^{\mathcal{E}}_{\mathcal{N}} - X^{\mathcal{E}}_{\mathcal{N}}Y^{\mathcal{E}}_{\mathcal{E}\setminus\mathcal{N}}) + \sum_{\mathcal{F}\succ_{2,0}\mathcal{N}}\ \sum_{\mathcal{E}'\perp_{\mathcal{F}}\mathcal{E},\ \mathcal{N}\in\mathcal{E}'} \varepsilon(\mathcal{E}',\mathcal{N})(X^{\mathcal{E}'}_{\mathcal{N}}(Y^{\mathcal{E}}_{\mathcal{N}} - Y^{\mathcal{F}\setminus\mathcal{E}}_{\mathcal{E}'\setminus\mathcal{N}}) - Y^{\mathcal{E}'}_{\mathcal{N}}(X^{\mathcal{E}}_{\mathcal{N}} - X^{\mathcal{F}\setminus\mathcal{E}}_{\mathcal{E}'\setminus\mathcal{N}})). \tag{19.21}$$

Discussion 19.11. Let us calculate the discrete Lie derivative of the commutator on a regular quasic-cubical mesh and compare it to the continuum. Let $D \in \mathbb{N}^+$, $\mathcal{K}$ be a regular quasi-cubical of dimension D with unit orthogonal directions $e_1, ..., e_D \in \mathbb{R}^D$ and lengths $h_1, ..., h_D \in \mathbb{R}^+$. We assume that for any $p \in \{0, ..., D\}$, any p-element ordered set $I \subset (1, ..., D)$, and any p-cell with directions $e_{I_1}, ..., e_{I_p}$, its embedded orientation is induced by the multivector $e_{I_1} \wedge ... \wedge e_{I_p}$. For any $\alpha = (\alpha_1, ..., \alpha_D) \in \mathbb{N}^D$ by $\mathcal{N}_\alpha$ we will denote the node with coordinates $(\alpha_1 h_1, ..., \alpha_D h_D)$.
Let $X, Y \in \mathcal{X}K$. Let $p \in \{1, ..., D\}$, $\alpha = (\alpha_1, ..., \alpha_D) \in \mathbb{N}^D$. $s \in \{0,1\}$ and take the edge $\mathcal{E}_{p,\alpha}$ with endpoints $\mathcal{N}_\alpha$ and $\mathcal{N}_{\alpha+e_p}$. Then for $s \in 0,1$, assuming $\mathcal{N}_{\alpha+se_p}$ is an interior node,

$$\begin{aligned}[X,Y]^{\mathcal{E}_{p,\alpha}}_{\mathcal{N}_{\alpha+se_p}} &= 2(-1)^{1-s}(X^{\mathcal{E}_{p,\alpha}}_{\mathcal{N}_{\alpha+(1-s)e_p}}Y^{\mathcal{E}_{p,\alpha}}_{\mathcal{N}_{\alpha+se_p}} - Y^{\mathcal{E}_{p,\alpha}}_{\mathcal{N}_{\alpha+(1-s)e_p}}X^{\mathcal{E}_{p,\alpha}}_{\mathcal{N}_{\alpha+se_p}}) \\ &\quad + \sum_{q\neq p}\sum_{t\in\{-1,0\}} (-1)^{t+1} \\ &\qquad (X^{\mathcal{E}_{q,\alpha+se_p+te_q}}_{\mathcal{N}_{\alpha+se_p}}(Y^{\mathcal{E}_{p,\alpha}}_{\mathcal{N}_{\alpha+se_p}} - Y^{\mathcal{E}_{p,\alpha+(2t+1)e_q}}_{\mathcal{N}_{\alpha+(2t+1)e_q+se_p}}) - Y^{\mathcal{E}_{q,\alpha+se_p+te_q}}_{\mathcal{N}_{\alpha+se_p}}(X^{\mathcal{E}_{p,\alpha}}_{\mathcal{N}_{\alpha+se_p}} - X^{\mathcal{E}_{p,\alpha+(2t+1)e_q}}_{\mathcal{N}_{\alpha+(2t+1)e_q+se_p}})).\end{aligned} \tag{19.22}$$

Now, assume these discrete vector fields come from continuous ones:

$$X = J\mathbf{X} = J\left(\sum_{p=1}^{D} f^p \frac{\partial}{\partial x^p}\right),\ Y = J\mathbf{Y} = J\left(\sum_{p=1}^{D} g^p \frac{\partial}{\partial x^p}\right). \tag{19.23}$$

Then

$$\begin{aligned}[X,Y]^{\mathcal{E}_{p,\alpha}}_{\mathcal{N}_{\alpha+se_p}} &= (-1)^{1-s}\frac{f^p(\alpha+(1-s)h_pe_p)g^p(\alpha+sh_pe_p) - f^p(\alpha+sh_pe_p)g^p(\alpha+(1-s)h_pe_p)}{2h_p^2} \\ &\quad + \sum_{q\neq p}\sum_{t\in\{-1,0\}} (-1)^{t+1} \\ &\qquad \left(\frac{f^q(\alpha+sh_pe_p)}{2h_q}\left(\frac{g^p(\alpha+sh_pe_p) - g^p(\alpha+sh_pe_p+(2t+1)h_qe_q)}{2h_p}\right)\right. \\ &\qquad \left. - \frac{g^q(\alpha+sh_pe_p)}{2h_q}\left(\frac{f^p(\alpha+sh_pe_p) - f^p(\alpha+sh_pe_p+(2t+1)h_qe_q)}{2h_p}\right)\right).\end{aligned} \tag{19.24}$$

For simplicity, let $s = 0$. Then,

$$\begin{aligned}[X,Y]^{\mathcal{E}_{p,\alpha}}_{\mathcal{N}_\alpha} &= -\frac{f^p(\alpha+h_pe_p)g^p(\alpha) - f^p(\alpha)g^p(\alpha+h_pe_p)}{2h_p^2} \\ &\quad + \sum_{q\neq p}\sum_{t\in\{-1,0\}} (-1)^{t+1} \\ &\quad \left(\frac{f^q(\alpha)}{2h_q}\left(\frac{g^p(\alpha) - g^p(\alpha+(2t+1)h_qe_q)}{2h_p}\right) - \frac{g^q(\alpha)}{2h_q}\left(\frac{f^p(\alpha) - f^p(\alpha+(2t+1)h_qe_q)}{2h_p}\right)\right).\end{aligned} \tag{19.25}$$

Hence,

$$
\begin{aligned}
[X,Y]^{\mathcal{E}_{p,\alpha}}_{\mathcal{N}_\alpha} = & \frac{f^p(\alpha)g^p(\alpha+h_pe_p)-f^p(\alpha+h_pe_p)g^p(\alpha)}{2h_p^2} \\
& + \sum_{q\neq p}\left(\frac{f^q(\alpha)}{2h_q}\left(\frac{g^p(\alpha+h_qe_q)-g^p(\alpha)}{2h_p}\right)-\frac{g^q(\alpha)}{2h_q}\left(\frac{f^p(\alpha+h_qe_q)-f^p(\alpha)}{2h_p}\right)\right). \\
& + \sum_{q\neq p}\left(\frac{f^q(\alpha)}{2h_q}\left(\frac{g^p(\alpha)-g^p(\alpha-h_qe_q)}{2h_p}\right)-\frac{g^q(\alpha)}{2h_q}\left(\frac{f^p(\alpha)-f^p(\alpha-h_qe_q)}{2h_p}\right)\right)
\end{aligned}
\tag{19.26}
$$

Up to a first order approximation, the last expression equals to

$$
\begin{aligned}
[X,Y]^{\mathcal{E}_{p,\alpha}}_{\mathcal{N}_\alpha} \approx & \frac{\left(f^p\frac{\partial g^p}{\partial x^p}-g^p\frac{\partial f^p}{\partial x^p}\right)(\alpha)}{2h_p} \\
& + \sum_{q\neq p}\frac{\left(f^q\frac{\partial g^p}{\partial x^q}-g^q\frac{\partial f^p}{\partial x^q}\right)(\alpha)}{4h_p} \\
& + \sum_{q\neq p}\frac{\left(f^q\frac{\partial g^p}{\partial x^q}-g^q\frac{\partial f^p}{\partial x^q}\right)(\alpha)}{4h_p},
\end{aligned}
\tag{19.27}
$$

which simplifies to

$$
[X,Y]^{\mathcal{E}_{p,\alpha}}_{\mathcal{N}_\alpha} \approx \sum_{q=1}^{D}\frac{\left(f^q\frac{\partial g^p}{\partial x^q}-g^q\frac{\partial f^p}{\partial x^q}\right)(\alpha)}{2h_p}. \tag{19.28}
$$

If we denote $\mathbf{Z} := [\mathbf{X},\mathbf{Y}]$, its expression in coordinates is

$$
\mathbf{Z} = \sum_{p=1}^{D} z^p\frac{\partial}{\partial x^p}, \tag{19.29}
$$

$$
z^p = \sum_{q=1}^{D}(f^q\frac{\partial g^p}{\partial x^q}-g^q\frac{\partial f^p}{\partial x^q}). \tag{19.30}
$$

Hence, the discrete Lie bracket further simplifies to

$$
[X,Y]^{\mathcal{E}_{p,\alpha}}_{\mathcal{N}_\alpha} \approx \frac{z^p(\alpha)}{2h_p}. \tag{19.31}
$$

However, by definition, the right hand side expresses the coefficients of the discrete vector field that approximates $\mathbf{Z}$. Hence,

$$
J([\mathbf{X},\mathbf{Y}]) \approx [J\mathbf{X}, J\mathbf{Y}], \tag{19.32}
$$

where the left bracket is the continuum Lie bracket, while the right bracket is the discrete Lie bracket. This calculation is the reason for the definition we gave earlier.

20 Discrete vector bundles and covariant exterior derivative

Definition 20.1. Let $d \in \mathbb{N}$, K be a quasi-cubical mesh of dimension d, V be a vector space, $p \in \{0,...,d\}$. The **space of V-valued p-cochains in** K $C^p(K,V)$ is defined by

$$
C^p(K,V) := \mathrm{Hom}(C_pK,V) \cong V \otimes C^pK. \tag{20.1}
$$

The **space of V-valued cochains in** K $C^\bullet(K,V)$ is defined by

$$
C^\bullet(K,V) := \bigoplus_{p=0}^{d} C^p(K,V). \tag{20.2}
$$

Definition 20.2. Let K be a quasi-cubical mesh, V be a vector space, $p \in \mathbb{N}$, $q \in \mathbb{N}$. Define the **cup product of a vector-valued cochain with a cochain**

$$\underline{\smile} \colon C^p(K,V) \times C^q K \to C^{p+q}(K,V) \tag{20.3}$$

as follows: for any $v \in V$, $\tau^p \in C^p K$, $\sigma^q \in C^q K$,

$$(v \otimes \tau^p) \underline{\smile} \sigma^q := v \otimes (\tau^p \smile \sigma^q), \tag{20.4}$$

and extend it by linearity on $C^p(K,V) \times C^q K$.

Remark 20.3. Let K be a quasi-cubical mesh, V be a vector space, $v \in V$, $p \in \mathbb{N}$, $\sigma^p \in C^p K$. Denote by 1 the identity zero-cochain on K. Then

$$v \otimes \sigma^p = v \otimes (1 \smile \sigma^p) = (v \otimes 1) \underline{\smile} \sigma^p. \tag{20.5}$$

Abuse the notation and identify $v \in V$ with $v \otimes 1 \in C^0(K,V)$. Then Equation (20.5) reads as

$$v \otimes \sigma^p = v \underline{\smile} \sigma^p. \tag{20.6}$$

Definition 20.4. Let $d \in \mathbb{N}$ K be a quasi-cubical mesh of dimension d, V be a vector space, $p \in \{0, ..., d-1\}$. A **discrete covariant exterior derivative on vector-valued p-cochains** is a linear map $\nabla_p \colon C^p(K,V) \to C^{p+1}(K,V)$ with the Leibniz property: for any $\sigma_V^0 \in C^0(K,V)$, $\tau^p \in C^p K$,

$$\nabla(\sigma_V^0 \underline{\smile} \tau^p) = \nabla \sigma_V^0 \underline{\smile} \tau^p + \sigma_V^0 \underline{\smile} \delta \tau^p. \tag{20.7}$$

The corresponding **discrete covariant exterior derivative on vector-valued cochains** $\nabla \colon C^\bullet(K,V) \to C^\bullet(K,V)$ is then ∇_p when acting on $C^p(K,V)$ $(p = 0, ..., d-1)$ and zero on $C^d(K,V)$.

Proposition 20.5. Let K be a quasi-cubical mesh, V be a vector space, $v \in V$, $p \in \mathbb{N}$, $\sigma^p \in C^p K$. Then

$$\nabla_p(v \underline{\smile} \sigma^p) = \nabla_0 v \underline{\smile} \sigma^p + v \underline{\smile} \delta_p \sigma^p. \tag{20.8}$$

Proof.

$$\nabla_p(v \underline{\smile} \sigma^p) := \nabla_p(v \otimes \sigma^p) = \nabla_p((v \otimes 1) \underline{\smile} \sigma^p) = \nabla_0 (v \otimes 1) \underline{\smile} \sigma^p + (v \otimes 1) \underline{\smile} \delta_p \sigma^p =: \nabla_0 v \underline{\smile} \sigma^p + v \underline{\smile} \delta_p \sigma^p. \tag{20.9}$$

□

Proposition 20.6. Let K be a quasi-cubical mesh, V be a vector space, $\nabla \colon C^\bullet(K,V) \to C^\bullet(K,V)$ be a discrete exterior covariant derivative. Then ∇ satisfies the graded Leibniz rule: for any $p, q \in \mathbb{N}$, $\sigma_V^p \in C^p(K,V)$, $\tau^q \in C^q K$,

$$\nabla_{p+q}(\sigma_V^p \underline{\smile} \tau^q) = \nabla_p \sigma_V^p \underline{\smile} \tau^q + (-1)^p \sigma_V^p \underline{\smile} \delta_q \tau^q. \tag{20.10}$$

Proof. It is enough to prove the proposition for a product element σ_V^p. Let $v \in V$, $\theta^p \in C^p K$ and $\sigma_V^p = v \underline{\smile} \theta^p$. Then

$$\begin{aligned}
\nabla_{p+q}(\sigma_V^p \underline{\smile} \tau^q) &= \nabla_{p+q}(v \underline{\smile} (\theta^p \smile \tau^q)) \\
&= \nabla_0 v \underline{\smile} (\theta^p \smile \tau^q) + v \underline{\smile} (\delta_p \theta^p \smile \tau^q) + (-1)^p v \underline{\smile} (\theta^p \smile \delta_q \tau^q) \\
&= \nabla_p (v \underline{\smile} \theta^p) \underline{\smile} \tau^q + (-1)^p (v \underline{\smile} \theta^p) \underline{\smile} \delta_q \tau^q \\
&= \nabla_p \sigma_V^p \underline{\smile} \tau^q + (-1)^p \sigma_V^p \underline{\smile} \delta_q \tau^q. \quad \square
\end{aligned} \tag{20.11}$$

Part IV
Physics

21 Continuous heat transport

Discussion 21.1. In this section we will consider the heat transport phenomenon in both transient and steady-state form. Our formulation will be represented in the language of differential forms because they better represent the meaning of physical quantities. Various (weak) reformulations will be presented – those reformulations will give us hints on how to construct purely discrete formulations.
We will formulate our phenomenon in arbitrary dimensions, although our model problem is in physical 3-dimensional space. The reason is that we conduct tests in different dimensions and, also, transport phenomena can be applied in domains with different dimensions.

Discussion 21.2. Let:

- D be a positive integer (space dimension);
- X be an open region in $\mathbb{R}^D$ (the space region);
- $t_0[T] \in \mathbb{R}$ be the initial time;
- $I := [t_0, \infty)$;

The main physical quantities in our model are:

- **temperature** $u[\Theta] \colon I \to \Omega^0 X$ characterised as follows: for any moment $t \in I$ and any point $x \in X$,

$$\text{“temperature } [\Theta] \text{ on } x \text{ at time } t\text{”} = u(t)(x) := u(t, x); \tag{21.1}$$

- heat energy $Q[E] \colon I \to \Omega^D X$: for any moment $t \in I$ and any volume $V_D \subseteq X$,

$$\text{“total heat energy } [E] \text{ of the system on } V \text{ at time } t\text{”} = \int_{V_D} Q(t); \tag{21.2}$$

- **heat flow rate** $q[ET^{-1}] \colon I \to \Omega^{D-1} X$ characterised as follows: for any time interval $[t_1, t_2] \subset I$ and any hypersurface $S_{D-1} \subset X$,

$$\text{“total flow } [E] \text{ through } S_{D-1} \text{ in } [t_1, t_2]\text{”} = \int_{t_1}^{t_2} \left(\int_{S_{D-1}} q(t) \right) dt. \tag{21.3}$$

  (Here we assume that S_{D-1} is oriented. Let U_D and V_D be adjacent regions having S_{D-1} as a common boundary, such that $\varepsilon(U_D, S_{D-1}) = -1$, $\varepsilon(V_D, S_{D-1}) = 1$. Then the above equation measures the total flow from V_D to U_D.)

We will also need the dual variables of heat energy, temperature, and flow rate.

- **dual temperature** $\tilde{u}[\Theta L^D] \colon I \to \Omega^D X$ defined by

$$\tilde{u} := \star_0 u \tag{21.4}$$

  (althiugh using non-zero based temperature scale might make $\star_0$ not well defined, this will not cause problems as we will always take temperature differences when substituting in equations);

- **heat energy density** $\tilde{Q}[EL^{-D}] \colon I \to \Omega^0 X$ defined by

$$\tilde{Q} := \star_D Q; \tag{21.5}$$

- **dual flow rate** $\tilde{q}[EL^{1-D}] \colon I \to \Omega^1 X$ defined by

$$\tilde{q} := \star_1^{-1} q = (-1)^{D-1} q; \tag{21.6}$$

The governing laws are formulated as follows.

- Let $f[ET^{-1}]\colon I \to \Omega^D X$ be the **internal production rate**, characterised as follows: for any time interval $[t_1, t_2] \subset I$ and any volume $V_D \subseteq X$,

$$\text{“total net heat production } [E] \text{ in } V_D \text{ in } [t_1,t_2]\text{”} = \int_{t_1}^{t_2} \left(\int_{V_D} f(t) \right) dt. \tag{21.7}$$

  Its dual is the **internal production rate density** $\tilde{f}[EL^{-D}T^{-1}]\colon I \to \Omega^0 X$ defined by

$$\tilde{f} = \star_D f. \tag{21.8}$$

- **Conservation of heat energy** is given by the following relation: for any time interval $[t_1, t_2] \subset I$ and any volume $V_D \subseteq X$,

$$\begin{aligned}\text{“heat difference on } V_D \text{ between moments } t_2 \text{ and } t_1\text{”} = &- \text{“heat outflow through the boundary of } V_D \text{ in } [t_1,t_2]\text{”} \\ &+ \text{“heat production inside } V_D \text{ in } [t_1,t_2]\text{”}.\end{aligned} \tag{21.9}$$

  In symbolic terms, the last equation is written as

$$\int_{V_D} (Q(t_2) - Q(t_1)) = - \int_{t_1}^{t_2} \left(\int_{\partial V_D} q(t) \right) dt + \int_{t_1}^{t_2} \left(\int_{V_D} f(t) \right) dt. \tag{21.10}$$

  Using Stokes' theorem twice, we get the equation

$$\int_{t_1}^{t_2} \left(\int_{V_D} \frac{\partial Q}{\partial t} \right) dt = - \int_{t_1}^{t_2} \left(\int_{V_D} d_{D-1} q \right) dt + \int_{t_1}^{t_2} \left(\int_{V_D} f \right) dt. \tag{21.11}$$

  Since the time interval $[t_1, t_2]$ and the volume V_D are arbitrary, we can drop integrals and arrive at the differential form

$$\frac{\partial Q}{\partial t} = -d_{D-1} q + f. \tag{21.12}$$

- Let $u_0[\Theta] \in \Omega^0 X$ be the initial temperature. The **initial condition** is the prescribed initial temperature:

$$u(t_0) = u_0. \tag{21.13}$$

- Let $\tilde{\pi}[EL^{-D}\Theta^{-1}]\colon \Omega^0 X \to \Omega^0 X$ be the **dual volumetric heat capacity**. The **relation between temperature change and heat energy change** is given by

$$\frac{\partial Q}{\partial t} = \star_0 \left(\frac{\partial \tilde{Q}}{\partial t} \right) = \star_0 \left(\tilde{\pi} \frac{\partial u}{\partial t} \right). \tag{21.14}$$

  The **volumetric heat capacity** $\pi[EL^{-D}\Theta^{-1}]\colon \Omega^D X \to \Omega^D X$ is related to $\tilde{\pi}$ by

$$\pi = \star_0 \circ \tilde{\pi} \circ \star_0^{-1} = \star_0 \circ \tilde{\pi} \circ \star_D. \tag{21.15}$$

- Consider two adjacent volumes U_D and V_D with a common surface S_{D-1}, such that $\varepsilon(U_D, S_{D-1}) = -1$ and $\varepsilon(V_D, S_{D-1}) = 1$. According to the second law of thermodynamics, heat flows from regions of higher temperature to regions of lower temperatures. Therefore, the net flow through S_{D-1} is in the negative direction of the temperature difference between U_D and V_D.

  Let $\kappa[EL^{2-D}T^{-1}\Theta^{-1}]\colon \Omega^{D-1} X \to \Omega^{D-1} X$ be the **thermal conductivity**. The **Fourier's constitutive relation** quantifies the above relation by using κ as a proportionality factor:

$$q = \kappa d_D^\star \tilde{u} = \kappa d_D^\star \star_0 u = -\kappa \star_1 d_0 u = - \star_1 \tilde{\kappa} d_0 u, \tag{21.16}$$

  where we have denoted the **dual conductivity**

$$\tilde{\kappa} := \star_1^{-1} \kappa \star_1 \, [EL^{2-D}T^{-1}\Theta^{-1}]\colon \Omega^1 X \to \Omega^1 X. \tag{21.17}$$

We complete our model with boundary conditions. Let Γ_D, Γ_N form a partition of ∂X into Dirichlet and Neumann boundary.

- Let $g_D[\Theta] \colon I \to \Omega^0\Gamma_D$ be the prescribed temperature on the Dirichlet boundary Γ_D. The **Dirichlet boundary condition** is given by

$$\mathrm{tr}_{\Gamma_D,0}\, u := u|_{\Gamma_D} = g_D. \tag{21.18}$$

- Let $g_N[ET^{-1}] \colon I \to \Omega^{D-1}\Gamma_N$ be the prescribed flow rate on the Neumann boundary Γ_N. The **Neumann boundary condition** is given by

$$\mathrm{tr}_{\Gamma_N,D-1}\, q = g_N. \tag{21.19}$$

Its dual is the **density of flow rate** $\widetilde{g_N}[EL^{1-D}T^{-1}] \colon I \to \Omega^0\Gamma_N$,

$$\widetilde{g_N} = \star_{\Gamma_N,D-1} g_N \tag{21.20}$$

21.1 Primal strong formulation

21.1.1 Transient

Formulation 21.3. The **strong differential formulation for heat transport** is obtained by representing heat energy and heat flow rate in terms of temperature.
Let:

- D be a positive integer (space dimension);
- X be an open region in $\mathbb{R}^D$ (the space region);
- $t_0[T] \in \mathbb{R}$ be the initial time;
- $I := [t_0, \infty)$;
- $u_0[\Theta] \in \Omega^0 X$ be the initial temperature;
- $\tilde{f}[EL^{-D}T^{-1}] \in \Omega^0 X$ be the dual internal production rate;
- Γ_D, Γ_N form a partition of ∂X;
- $g_D[\Theta] \colon I \to \Omega^0\Gamma_D$ be the prescribed temperature on the Dirchlet boundary.
- $\widetilde{g_N}[ET^{-1}] \colon I \to \Omega^0\Gamma_N$ be the prescribed flow rate density through the Neumann boundary;
- $\tilde{\pi}[EL^{-D}\Theta^{-1}] \colon \Omega^0 X \to \Omega^0 X$ be the dual volumetric heat capacity;
- $\tilde{\kappa}[EL^{-1}T^{-1}\Theta^{-1}] \colon \Omega^1 X \to \Omega^1 X$ be the dual thermal conductivity.

We are solving the following problem for the unknown temperature $u[\Theta] \colon I \to \Omega^0 X$.

$$\tilde{\pi}\frac{\partial u}{\partial t} = -(d_1^\star \circ \tilde{\kappa} \circ d_0)u + \tilde{f} \qquad [EL^{-D}T^{-1}], \tag{21.21a}$$

$$\mathrm{tr}_{\Gamma_D,0}\, u = g_D \qquad [\Theta], \tag{21.21b}$$

$$-(\star_{\Gamma_N,D-1} \circ \mathrm{tr}_{\Gamma_N,D-1} \circ \star_1 \circ \tilde{\kappa} \circ d_0)u = \widetilde{g_N} \qquad [EL^{1-D}T^{-1}], \tag{21.21c}$$

$$u(t_0,\cdot) = u_0 \qquad [\Theta]. \tag{21.21d}$$

21.1.2 Steady-state

Formulation 21.4. [Primal strong formulation for the steady-state continuous heat equation using differential forms] Let:

- D be a positive integer (space dimension);
- X be an open region in $\mathbb{R}^D$ (the space region);
- $\tilde{f}[EL^{-D}T^{-1}] \in \Omega^0 X$ be the dual internal production rate;
- Γ_D, Γ_N form a partition of ∂X;
- $g_D[\Theta] \in \Omega^0\Gamma_D$ be the prescribed temperature on the Dirchlet boundary.

- $\widetilde{g_N}[EL^{1-D}T^{-1}] \in \Omega^0\Gamma_N$ be the prescribed flow rate density through the Neumann boundary;
- $\tilde{\kappa}[EL^{-1}T^{-1}\Theta^{-1}]\colon \Omega^1 X \to \Omega^1 X$ be the dual thermal conductivity.

We are solving the following problem for the unknown temperature $u[\Theta] \in \Omega^0 X$:

$$(d_1^\star \circ \tilde{\kappa} \circ d_0)u = \tilde{f} \qquad [EL^{-D}T^{-1}], \tag{21.22a}$$

$$\operatorname{tr}_{\Gamma_D,0} u = g_D \qquad [\Theta], \tag{21.22b}$$

$$-(\star_{\Gamma_N,D-1} \circ \operatorname{tr}_{\Gamma_N,D-1} \circ \star_1 \circ \tilde{\kappa} \circ d_0)u = \widetilde{g_N} \qquad [EL^{1-D}T^{-1}]. \tag{21.22c}$$

21.2 Primal weak formulation

21.2.1 Transient

Discussion 21.5. Using the variables from Formulation 21.3 we are going to introduce an alternative (primal weak) formulation. Let $w \in \operatorname{Ker}\operatorname{tr}_{\Gamma_D,0}$ be a test function (later on the differentiablity assumptions on w can be weakened). Multiply the conservation of energy with w and integrate over X:

$$\begin{aligned}
\int_X w \wedge \frac{\partial Q}{\partial t} &= -\int_X (w \wedge d_{D-1}q) + \int_X (w \wedge f) \\
&= -\int_{\partial X} \operatorname{tr}_{\partial X,D-1}(w \wedge q) + \int_X (d_0 w \wedge q) + \int_X (w \wedge f) \\
&= -\int_{\Gamma_N} (\operatorname{tr}_{\Gamma_N,0} w \wedge \operatorname{tr}_{\Gamma_N,D-1} q) - \int_X (d_0 w \wedge \star_1 \tilde{\kappa} d_0 u) + \int_X (w \wedge f) \\
&= -\int_{\Gamma_N} (\operatorname{tr}_{\Gamma_N,0} w \wedge g_N) - \int_X (d_0 w \wedge \star_1 \tilde{\kappa} d_0 u) + \int_X (w \wedge f) \\
&= -\int_{\Gamma_N} (\operatorname{tr}_{\Gamma_N,0} w \wedge g_N) - \langle d_0 w, \tilde{\kappa} d_0 u\rangle_{X,1} + \int_X (w \wedge f).
\end{aligned} \tag{21.23}$$

We also have:

$$\int_X w \wedge \frac{\partial Q}{\partial t} = \int_X w \wedge \left(\star_0 \tilde{\pi} \frac{\partial u}{\partial t}\right) = \langle w, \tilde{\pi}\frac{\partial u}{\partial t}\rangle_{X,0}. \tag{21.24}$$

Equating both equations leads to the following (primal weak) formulation.

Formulation 21.6. [Primal weak formulation for the transient continuous heat equation with differential forms] Let:

- Let D be a positive integer (space dimension);
- X be an open region in $\mathbb{R}^D$ representing the material body;
- $t_0 \in \mathbb{R}$ be the initial time;
- $I = [t_0, \infty)$;
- $f[ET^{-1}]\colon I \to \Omega^D X$ be the internal production rate;
- $u_0[\Theta] \in \Omega^0 X$ be the initial temperature;
- $\tilde{\pi}[EL^{-D}\Theta^{-1}]\colon \Omega^0 X \to \Omega^0 X$ be the heat capacity of the material;
- $\tilde{\kappa}[EL^{2-D}T^{-1}\Theta^{-1}]\colon \Omega^1 X \to \Omega^1 X$ be the thermal conductivity of the material;
- $\partial X = \Gamma_D \cup \Gamma_N$ be the partition of the boundary of X into Dirichlet (Γ_D) and Neumann (Γ_N) regions;
- $g_D[\Theta]\colon I \to \Omega^0\Gamma_D$ be the prescribed temperature on the Dirichlet boundary;
- $g_N[ET^{-1}]\colon I \to \Omega^{D-1}\Gamma_N$ be the prescribed flow rate on the Neumann boundary.

Define the following operators:

$$A\colon \Omega^0 X \times (I \to \Omega^0 X) \to \mathbb{R}, \quad A(v,w) := \langle d_0 v, \tilde{\kappa} d_0 w\rangle_{X,0} \quad [ET^{-1}\Theta^{-1}], \tag{21.25a}$$

$$B\colon \Omega^0 X \times (I \to \Omega^0 X) \to \mathbb{R}, \quad B(v,w) := \langle v, \tilde{\pi} w\rangle_{X,0} \quad [E\Theta^{-1}], \tag{21.25b}$$

$$G\colon \Omega^0 X \to \mathbb{R}, \quad G(v) := \int_{\Gamma_N} (\operatorname{tr}_{\Gamma_N} v \wedge g_N) \quad [ET^{-1}], \tag{21.25c}$$

$$F\colon \Omega^0 X \to \mathbb{R}, \quad F(v) := \int_X (v \wedge f) \quad [ET^{-1}]. \tag{21.25d}$$

Our unknowns is temperature $u[\Theta]\colon I \to \Omega^0 X$. We are solving the following problem for u:

$$\forall v[\Theta] \in \operatorname{Ker} \operatorname{tr}_{\Gamma_D,0}, \quad B(v, \frac{\partial u}{\partial t}) + A(v,u) = F(v) - G(v) \quad [ET^{-1}\Theta], \tag{21.26a}$$

$$\operatorname{tr}_{\Gamma_D,0} u = g_D \quad [\Theta], \tag{21.26b}$$

$$u(t_0) = u_0 \quad [\Theta]. \tag{21.26c}$$

The flow rate $q[ET^{-1}]\colon I \to \Omega^{D-1} X$ is calculated by the formula

$$q(t,x) = \begin{cases} (-\star_1 \tilde{\kappa} d_0 u)(t,x), & x \notin \Gamma_N \\ g_N(t,x), & x \in \Gamma_N \end{cases}, \ t \in I. \tag{21.27}$$

21.2.2 Steady-state

Formulation 21.7. [Primal weak formulation for the steady-state continuous heat equation with differential forms] Let:

- D be a positive integer (space dimension);
- X be an open region in $\mathbb{R}^D$, representing a material body;
- $\tilde{\kappa}[EL^{-1}T^{-1}\Theta^{-1}]\colon \Omega^1 X \to \Omega^1 X$ be the thermal conductivity of the material;
- $f[ET^{-1}] \in \Omega^D X$ be the internal production rate;
- $\partial X = \Gamma_D \cup \Gamma_N$ be the partition of the boundary of X into Dirichlet (Γ_D) and Neumann (Γ_N) regions;
- $g_D[\Theta] \in \Omega^0 \Gamma_D$ be the prescribed temperature on the Dirichlet boundary;
- $g_N[ET^{-1}] \in \Omega^{D-1}\Gamma_N$ be the prescribed flow rate on the Neumann boundary.

Our unknowns is temperature $u[\Theta] \in \Omega^0 X$. We are solving the following problem for u:

$$\forall v \in \operatorname{Ker} \operatorname{tr}_{\Gamma_D,0}, \quad \langle d_0 v, \tilde{\kappa} d_0 u\rangle_{X,1} = \int_X (v \wedge f) - \int_{\Gamma_N} (\operatorname{tr}_{\Gamma_N,0} v \wedge g_N) \quad [ET^{-1}\Theta], \tag{21.28a}$$

$$\operatorname{tr}_{\Gamma_D,0} u = g_D \quad [\Theta]. \tag{21.28b}$$

The flow rate $q[ET^{-1}] \in \Omega^{D-1} X$ is calculated by the formula

$$q(x) := \begin{cases} (-\star_1 \tilde{\kappa} d_0 u)(x), & x \notin \Gamma_N \\ g_N(x), & x \in \Gamma_N \end{cases}. \tag{21.29}$$

21.3 Mixed weak formulation

21.3.1 Transient

Discussion 21.8. We are going to formulate the **mixed weak formulation for continuous heat transport with differential forms**. Consider the model Discussion 21.2 with the same domains and variable names. Let $r[ET^{-1}] \in \operatorname{Ker}(\operatorname{tr}_{\Gamma_N, D-1})$. Then

$$\kappa^{-1} q = d_\star^D \tilde{u} = -\star_1 d_0 u. \tag{21.30}$$

Hence,

$$\begin{aligned}\langle r, \kappa^{-1} q\rangle_{X,D-1} &= \langle r, -\star_1 d_0 u\rangle_{X,D-1} \\ &= -\int_X d_0 u \wedge r \\ &= -\int_{\partial X} \operatorname{tr}_{\partial X,0} u \wedge \operatorname{tr}_{\partial X,D-1} r + \int_X (u \wedge d_{D-1} r) \\ &= -\int_{\Gamma_D} g_D \wedge \operatorname{tr}_{\Gamma_D,D-1} r + \langle \star_0 u, d_{D-1} r\rangle \\ &= -\int_{\Gamma_D} g_D \wedge \operatorname{tr}_{\Gamma_D,D-1} r + \langle \tilde{u}, d_{D-1} r\rangle.\end{aligned} \tag{21.31}$$

Let $\tilde{w}[\Theta] \in \Omega^D X$. Taking the inner product of the conservation law with $\tilde{w}$ gives

$$\langle \pi \frac{\partial \tilde{u}}{\partial t}, \tilde{w}\rangle_{X,D} = -\langle d_{D-1} q, \tilde{w}\rangle_{X,D} + \langle f, \tilde{w}\rangle. \tag{21.32}$$

This leads to the following (mixed weak) formulation.

Formulation 21.9. [Mixed weak formulation for the transient continuous heat equation with differential forms] Let:

- D be a positive integer (space dimension);
- X be a D-dimensional open region, representing a material body;
- $t_0 \in \mathbb{R}$ be the initial time;
- $I = [t_0, \infty)$ be the time-interval where the process occurs;
- $f[ET^{-1}] \colon I \to \Omega^D X$ be the internal production rate;
- $u_0[\Theta] \in \Omega^0 X$ be the initial temperature;
- $\kappa[EL^{2-D}T^{-1}\Theta^{-1}] \colon \Omega^{D-1} X \to \Omega^{D-1} X$ be the thermal conductivity of the material;
- $\pi[EL^{-D}\Theta^{-1}] \colon \Omega^D X \to \Omega^D X$ be the heat capacity of the material;
- $\partial X = \Gamma_D \cup \Gamma_N$ be the partition of the boundary of X into Dirichlet (Γ_D) and Neumann (Γ_N) regions;
- $g_D[\Theta] \colon I \to \Omega^0 \Gamma_D$ be the prescribed temperature on the Dirichlet boundary;
- $g_N[ET^{-1}] \colon I \to \Omega^{D-1} \Gamma_N$ be the prescribed flow rate on the Neumann boundary.

Define the following operators:

$$A \colon \Omega^{D-1} X \times (I \to \Omega^{D-1} X) \to \mathbb{R}, \quad A(r, s) := \langle r, \kappa^{-1} s\rangle_{X,D-1} \quad [E^{-1} T \Theta], \tag{21.33a}$$

$$B \colon \Omega^D X \times (I \to \Omega^{D-1} X) \to \mathbb{R}, \quad B(\tilde{w}, r) := \langle d_{D-1} r, \tilde{w}\rangle_{X,D} \quad [L^{-D}], \tag{21.33b}$$

$$C \colon \Omega^D X \times (I \to \Omega^D X) \to \mathbb{R}, \quad C(\tilde{w}, w^D) := \langle \pi w^D, \tilde{w}\rangle_{X,D} \quad [EL^{-2D}\Theta^{-1}], \tag{21.33c}$$

$$G \colon \Omega^{D-1} X \to \mathbb{R}, \quad G(r) := \int_{\Gamma_D} (\operatorname{tr}_{\Gamma_D,D-1} r \wedge g_D)\ [\Theta], \tag{21.33d}$$

$$F \colon \Omega^D X \to \mathbb{R}, \quad F(\tilde{w}) := \langle f, \tilde{w}\rangle_{X,D} \quad [ET^{-1}L^{-D}]. \tag{21.33e}$$

Our unknowns are:

- $q[ET^{-1}] \colon I \to \Omega^{D-1} X$ (heat flow rate);
- $\tilde{u}[\Theta L^D] \colon I \to \Omega^D X$ (dual temperature).

We are solving the following problem for q and $\tilde{u}$:

$$\forall r[ET^{-1}] \in \operatorname{Ker} \operatorname{tr}_{\Gamma_N,D-1}, \quad A(r, q) - B^T(r, \tilde{u}) = -G(r) \quad [ET^{-1}\Theta], \tag{21.34a}$$

$$\forall \tilde{w}[\Theta L^D] \in \Omega^D X, \quad -B(\tilde{w}, q) - C(\tilde{w}, \frac{\partial \tilde{u}}{\partial t}) = -F(\tilde{w}) \quad [ET^{-1}\Theta], \tag{21.34b}$$

$$\operatorname{tr}_{\Gamma_N,D-1} q = g_N \quad [ET^{-1}], \tag{21.34c}$$

$$\tilde{u}(t_0) = \star_{X,0} u_0 \quad [\Theta L^D]. \tag{21.34d}$$

The temperature $u[\Theta]\colon I \to \Omega^0 X$ is calculated in the post-processing phase by the formula

$$u(t,x) := \begin{cases} u_0(x), & t = t_0 \\ (\star_D \tilde{u})(t,x), & t > t_0 \text{ and } x \notin \Gamma_D \\ g_D(t,x), & t_0 > 0 \text{ and } x \in \Gamma_D \end{cases}. \tag{21.35}$$

21.3.2 Steady-state

Formulation 21.10. [Mixed weak formulation for the steady-state continuous heat equation with differential forms] Let:

- D be a positive integer (space dimension);
- X be a D-dimensional open region, representing a material body;
- $f[ET^{-1}] \in \Omega^D X$ be the internal production rate;
- $\kappa[EL^{2-D}T^{-1}\Theta^{-1}]\colon \Omega^{D-1}X \to \Omega^{D-1}X$ be the thermal conductivity of the material;
- $\partial X = \Gamma_D \cup \Gamma_N$ be the partition of the boundary of X into Dirichlet (Γ_D) and Neumann (Γ_N) regions;
- $g_D[\Theta] \in \Omega^0 \Gamma_D$ be the prescribed temperature on the Dirichlet boundary;
- $g_N[ET^{-1}] \in \Omega^{D-1}\Gamma_N$ be the prescribed flow rate on the Neumann boundary.

Define the following operators:

$$\begin{aligned}
&A\colon \Omega^{D-1}X \times \Omega^{D-1}X \to \mathbb{R}, && A(r,s) := \langle r, \kappa^{-1} s\rangle_{X,D-1} && [E^{-1}T\Theta], && (21.36a)\\
&B\colon \Omega^{D}X \times \Omega^{D-1}X \to \mathbb{R}, && B(\tilde{w},r) := \langle d_{D-1} r, \tilde{w}\rangle_{X,D} && [L^{-D}], && (21.36b)\\
&G\colon \Omega^{D-1}X \to \mathbb{R}, && G(r) := \int_{\Gamma_D} (\operatorname{tr}_{\Gamma_D,D-1} r \wedge g_D) && [\Theta], && (21.36c)\\
&F\colon \Omega^{D}X \to \mathbb{R}, && F(\tilde{w}) := \langle f, \tilde{w}\rangle_{X,D} && [ET^{-1}L^{-D}]. && (21.36d)
\end{aligned}$$

Our unknowns are:

- $q[ET^{-1}] \in \Omega^{D-1}X$ (heat flow rate);
- $\tilde{u}[\Theta L^D] \in \Omega^D X$ (dual temperature).

We are solving the following problem for q and u:

$$\begin{aligned}
&\forall r[ET^{-1}] \in \operatorname{Ker}\operatorname{tr}_{\Gamma_N,D-1}, && A(r,q) - B^T(r,\tilde{u}) = -G(r) && [ET^{-1}\Theta], && (21.37a)\\
&\forall \tilde{w}[\Theta L^D] \in \Omega^D X, && -B(\tilde{w},q) = -F(\tilde{w}) && [ET^{-1}\Theta], && (21.37b)\\
& && \operatorname{tr}_{\Gamma_N,D-1} q = g_N && [ET^{-1}]. && (21.37c)
\end{aligned}$$

The temperature $u[\Theta] \in \Omega^0 X$ is calculated in the post-processing phase by the formula

$$u(x) := \begin{cases} (\star_D \tilde{u})(x), & x \notin \Gamma_D \\ g_D(x), & x \in \Gamma_D \end{cases}. \tag{21.38}$$

22 Discrete heat transport

Notation 22.1. Let S be a set, T be a subset of S, V be a real vector space, $u \in \operatorname{Hom}_{\mathbb{R}}(\operatorname{Free}_{\mathbb{R}}(S), V)$. We will denote by

$$u|_T \in \operatorname{Hom}_{\mathbb{R}}(\operatorname{Free}_{\mathbb{R}}(T), V) \tag{22.1}$$

the map defined in the same way of u but acting on formal linear combinations of the elements in T.

Notation 22.2. Let $D \in \mathbb{N}$, K be a flat mesh of dimension D, $c_0 \in C_0(\partial K)$ with corresponding point $x \in \mathbb{R}^D$. By $\mathbf{n}_{c_0}$ we will denote the exterior unit normal at x to K. When c_0 has more than one non-parallel adjacent hyperfaces (for instance, in 3D, it can lie on an edge or at a corner), we will take some average of the normals to those faces.

The easiest one is to sum all exterior unit normals and divide by the length of the sum. This is the approach taken in the software implementation.
When $\Gamma \subseteq (\partial K)_0$, we will understand $\mathbf{n}$ as a function from Γ to $\mathbb{R}^D$ or as a linear map in $\mathrm{Hom}(\mathrm{Free}_{\mathbb{R}}(\Gamma), \mathbb{R}^D)$.

Discussion 22.3. We are going to state the governing laws for the discrete heat transport phenomenon in the strong formulation.
Let:

- D be a positive integer (physical dimension);
- M be a manifold-like flat mesh of dimension D;
- K be the Forman subdivision of M;
- $t_0 \in \mathbb{R}$ be the initial time, $I = [t_0, \infty)$.

Physical quantities in our model are:

- temperature $u[\Theta] \colon I \to C^0 K$;
- heat energy density $\tilde{Q}[EL^{-D}] \colon I \to C^0 K$;
- dual heat flow rate $\tilde{q}[EL^{1-D}T^{-1}] \colon I \to C^1 K$;

The governing laws are the following.

- Let $K' := K \setminus \partial K$ be the interior of K, $\tilde{f}[EL^{-D}T^{-1}] \in C^0 K'$ be the dual internal production rate; **Conservation of heat energy** is modeled by the equation

$$\left. \frac{\partial \tilde{Q}}{\partial t} \right|_{K'_0} = (\delta_1^\star \tilde{q})|_{K'_0} + \tilde{f}. \tag{22.2}$$

- Let $u_0[\Theta] \in C^0 K$ be the initial temperature. The **initial condition** is given by prescribed initial temperature:

$$u(t_0, \cdot) = u_0. \tag{22.3}$$

- Let $\tilde{\pi}[EL^{-D}\Theta^{-1}] \colon C^0 K \to C^0 K$ be the dual volumetric heat capacity (its matrix in the standard basis is diagonal). The **relation between temperature change and heat energy change** is given by

$$\frac{\partial \tilde{Q}}{\partial t} = \tilde{\pi} \frac{\partial u}{\partial t}. \tag{22.4}$$

- Let $\tilde{\kappa}[EL^{2-D}T^{-1}\Theta^{-1}] \colon C^1 K \to C^1 K$ be the dual thermal conductivity (its matrix in the standard basis is diagonal). The **Fourier's constitutive relation** is given by

$$\tilde{q} = -\tilde{\kappa}(\delta_0 u). \tag{22.5}$$

We complete our model with boundary conditions. Let Γ_D, Γ_N form a partition of ∂K into Dirichlet and Neumann boundary.

- Let $g_D[\Theta] \colon I \to C^0 \Gamma_D$ be the prescribed temperature on the Dirichlet boundary Γ_D. The **Dirichlet boundary condition** is given by

$$u|_{\Gamma_D} = g_D. \tag{22.6}$$

- Let $\mathbf{n}[L^{-1}] \colon \Gamma_N \to \mathbb{R}^d$ be the generalized exterior unit normal, $\widetilde{g_N}[EL^{1-D}T^{-1}] \colon I \to C^0 \Gamma_N$ of physical dimension $[EL^{1-D}T^{-1}]$ be the prescribed dual flow rate through the Neumann boundary. The **Neumann boundary condition** is given by

$$\overline{\tilde{q}}|_{\Gamma_N} \cdot \mathbf{n} = \widetilde{g_N}. \tag{22.7}$$

22.1 Primal strong formulation

22.1.1 Steady-state

Formulation 22.4. [Primal strong formulation for the steady-state discrete heat equation with discrete differential forms] Let:

- $D \in \mathbb{N}$;
- M be a manifold-like flat mesh of dimension D;
- K be the Forman subdivision of M;
- $K' := K \setminus \partial K$ be the interior of K;
- $\tilde{f}[EL^{-D}T^{-1}] \in C^0 K$ be the dual internal production rate;
- Γ_D, Γ_N form a partition of $(\partial K)_0$;
- $\mathbf{n}[L^{-1}]\colon \Gamma_N \to \mathbb{R}^D$ be the generalized exterior unit normal;
- $g_D[\Theta] \in C^0 \Gamma_D$ be the prescribed temperature on the Dirichlet boundary;
- $\widetilde{g_N}[EL^{1-D}T^{-1}] \in C^0 \Gamma_N$ be the prescribed flow rate density through the Neumann Boundary;
- $\tilde{\pi}[EL^{-D}\Theta^{-1}]\colon C^0 K \to C^0 K$ be the dual heat capacity (its matrix in the standard basis is diagonal);
- $\tilde{\kappa}[EL^{-D}T^{-1}\Theta^{-1}]\colon C^1 K \to C^1 K$ be the dual conductivity (its matrix in the standard basis is diagonal);

We are solving the following problem for $u[\Theta] \in C^0 K$:

$$(\operatorname{tr}_{K'_0} \circ \delta_1^\star \circ \tilde{\kappa} \circ \delta_0)u = \operatorname{tr}_{K'_0} \tilde{f} \quad \text{(balance of heat energy)} \quad [EL^{-D}T^{-1}], \tag{22.8a}$$

$$\operatorname{tr}_{\Gamma_D,0} u = g_D \quad \text{(Dirichlet boundary condition)} \quad [\Theta], \tag{22.8b}$$

$$-\left.\overline{(\tilde{\kappa} \circ \delta_0)u}\right|_{\Gamma_N} \cdot \mathbf{n} = \widetilde{g_N} \quad \text{(Neumann boundary condition)} \quad [EL^{1-D}T^{-1}]. \tag{22.8c}$$

22.1.2 Transient

Formulation 22.5. By substituting q with $\tilde{\kappa}\delta_0 u$ and Q with $\tilde{\pi}u$, we arrive at the following formulation with only one unknown (the temperature u). Let:

- $D \in \mathbb{N}$;
- M be a manifold-like flat mesh of dimension D;
- K be the Forman subdivision of M;
- $K' := K \setminus \partial K$ be the interior of K;
- $t_0 \in \mathbb{R}$ be the initial time, t_0 $[T]$;
- $I := [t_0, \infty)$ be the time interval;
- $u_0[\Theta] \in C^0 K$ be the initial temperature;
- $\tilde{f}[EL^{-D}T^{-1}] \in C^0 K$ be the dual internal production rate;
- Γ_D, Γ_N form a partition of $(\partial K)_0$;
- $\mathbf{n}[L^{-1}]\colon \Gamma_N \to \mathbb{R}^d$ be the generalized exterior unit normal;
- $g_D[\Theta]\colon I \to C^0 \Gamma_D$ be the prescribed temperature on the Dirichlet boundary;
- $\widetilde{g_N}[EL^{1-D}T^{-1}]\colon I \to C^0 \Gamma_N$ be the prescribed flow rate density on the Neumann boundary;
- $\tilde{\pi}[EL^{-D}\Theta^{-1}]\colon C^0 K \to C^0 K$ be a material property (dual heat capacity) of the nodes of K (its matrix in the standard basis is diagonal);
- $\tilde{\kappa}[EL^{2-D}T^{-1}\Theta^{-1}]\colon C^1 K \to C^1 K$ be a material property (dual thermal conductivity) of the edges of K (its matrix in the standard basis is diagonal).

We are solving the following problem for $u[\Theta] \colon I \to C^0 K$:

$$\operatorname{tr}_{K_0'}\left(\frac{\partial(\tilde{\pi} u)}{\partial t}\right) = \operatorname{tr}_{K_0'}(\tilde{f} - (\delta_1^\star \circ \tilde{\kappa} \circ \delta_0) u) \quad \text{(conservation of heat energy)} \quad [E L^{-D} T^{-1}], \tag{22.9a}$$

$$\operatorname{tr}_{\Gamma_D, 0} u = g_D \quad \text{(Dirichlet boundary condition)} \quad [\Theta], \tag{22.9b}$$

$$-\left.\overline{(\tilde{\kappa} \circ \delta_0) u}\right|_{\Gamma_N} \cdot \mathbf{n} = \widetilde{g_N} \quad \text{(Neumann boundary condition)} \quad [E L^{1-D} T^{-1}]. \tag{22.9c}$$

Discussion 22.6. Consider Formulation 22.5. This formulation is discrete in space but continuous in time. In order to numerically solve it we need to discretize the time variable. We will use the trapezoidal (Crank-Nicolson) method.
Let $\tau[T] \in \mathbb{R}^+$ be the time step, $s \in \mathbb{N}$, $t_s := t_0 + s\tau$, $u^s[\Theta] := u(t_s, \cdot) \in C^0 K$, $B[E L^{-D} T^{-1} \Theta^{-1}] := \delta_1^\star \circ \tilde{\kappa} \circ \delta_0$. Integrating the conservation of heat energy in $[t_s, t_{s+1}]$, we get

$$(\tilde{\pi} u^{s+1} - \tilde{\pi} u^s)|_{K_0'} = -\int_{t_s}^{t_{s+1}} (B u(t, \cdot))|_{K_0'} \, dt + \int_{t_s}^{t_{s+1}} \tilde{f} \, dt \approx -\frac{\tau}{2} (B u^s + B u^{s+1})|_{K_0'} + \tau \tilde{f}. \tag{22.10}$$

Rearranging, we get the discretized equation

$$\left.\left((\tilde{\pi} + \frac{\tau}{2} B) u^{s+1}\right)\right|_{K_0'} = \left.\left((\tilde{\pi} - \frac{\tau}{2} B) u^s\right)\right|_{K_0'} + \tau \tilde{f}. \tag{22.11}$$

Define the operators $L_\tau, R_\tau[E L^{-D} \Theta^{-1}] \in \operatorname{Hom}(C^0 K, C^0 K)$ by

$$L_\tau := \tilde{\pi} + \frac{\tau}{2} B, \tag{22.12a}$$

$$R_\tau := \tilde{\pi} - \frac{\tau}{2} B. \tag{22.12b}$$

The discretized in time (space is already discrete) temperature $\{u^s[\Theta] \in C^0 K\}_{s=0}^\infty$ is find iteratively as follows. $u^0 = u_0$ and for any $s > 0$, u^s is solution to the following problem:

$$(L_\tau u^s)|_{K_0'} = (R_\tau y_{s-1})|_{K_0'} + \tau \tilde{f} \quad \text{(balance of heat energy)} \quad [E L^{-D}], \tag{22.13a}$$

$$u^s|_{\Gamma_D} = g_D \quad \text{(Dirichlet boundary condition)} \quad [\Theta], \tag{22.13b}$$

$$-\left.\overline{(\tilde{\kappa} \circ \delta_0) u^s}\right|_{\Gamma_N} \cdot \mathbf{n} = \widetilde{g_N} \quad \text{(Neumann boundary condition)} \quad [E L^{1-D} T^{-1}]. \tag{22.13c}$$

Of course, in practice we solve it for a finite number of time steps. Usually, we compare tho adjacent solutions u^s and u^{s+1} and stop when the relative error is sufficiently small, i.e., until we reach a steady state.

22.2 Primal weak formulation

22.2.1 Steady-state

Formulation 22.7. [Primal weak formulation for the steady-state discrete heat equation with discrete differential forms] The following formulation is a discrete version of Formulation 21.7. Let:

- Let D be a positive integer (space dimension);
- K be an oriented quasi-cubical mesh of dimension D representing the material body;
- $[K]$ be the fundamental class of K;
- $\tilde{\kappa}[E L^{2-D} T^{-1} \Theta^{-1}] \colon C^1 K \to C^1 K$ be the thermal conductivity of the material, such that for any edge $c_1 \in K_1$, there exists some $\lambda > 0$ such that $\tilde{\kappa}(c^1) = \lambda c^1$;
- $f[E T^{-1}] \in C^D K$ be the internal production rate;
- $\partial K = \Gamma_D \cup \Gamma_N$ be the partition of the boundary of K into Dirichlet (Γ_D) and Neumann (Γ_N) regions;
- $[\Gamma_N]$ be the fundamental class of Γ_N, where Γ_N has the boundary orientation induced from K;
- $g_D[\Theta] \in C^0 \Gamma_D$ be the prescribed temperature on the Dirichlet boundary;
- $g_N[E T^{-1}] \in C^{D-1} \Gamma_N$ be the prescribed flow rate on the Neumann boundary.

Our unknown is temperature $u[\Theta] \in C^0 K$. We are solving the following problem for u:

$$\forall v \in \operatorname{Ker} \operatorname{tr}_{\Gamma_D,0},\ \langle \delta_0 v, \tilde{\kappa} \delta_0 u \rangle_{K,1} = (v \smile f)[K] - (\operatorname{tr}_{\Gamma_N,0} v \smile g_N)[\Gamma_N] \qquad [ET^{-1}\Theta], \tag{22.14a}$$

$$\operatorname{tr}_{\Gamma_D,0} u = g_D \qquad [\Theta]. \tag{22.14b}$$

The flow rate $q[ET^{-1}] \in C^{D-1} K$ is calculated in the post-processing phase by the formula

$$q(c_{D-1}) := \begin{cases} (-\star_1 \tilde{\kappa} \delta_0 u)(c_{D-1}), & c_{D-1} \in K_{D-1} \setminus (\Gamma_N)_{D-1} \\ g_N(c_{D-1}), & c_{D-1} \in (\Gamma_N)_{D-1} \end{cases}. \tag{22.15}$$

Discussion 22.8. We are going to derive a solution to Formulation 22.7. For any $p \in \{0, ..., D\}$ denote

$$n_p := |K_p| = \dim(C_p K) = \dim(C^p K). \tag{22.16}$$

The cochains $(N^0, ..., N^{n_0 - 1})$ form the standard basis of $C^0 K$. Define the matrix $\mathbf{A} \in M_{n_0 \times n_0}(\mathbb{R})$ by

$$\mathbf{A}_{i,j} := \langle \delta N^i, \tilde{\kappa} \delta_0 N^i \rangle_{K,1},\ i, j = 0, ..., n_0 - 1, \tag{22.17}$$

and the vectors $\mathbf{F}, \mathbf{G}, \mathbf{H} \in \mathbb{R}^{n_0}$ by

$$\mathbf{F}_i := (N^i \smile f)[K], \qquad i = 0, ..., n_0 - 1, \tag{22.18a}$$

$$\mathbf{G}_i := (\operatorname{tr}_{\Gamma_N,0} N^i \smile g_N)[\Gamma_N],\ i = 0, ..., n_0 - 1, \tag{22.18b}$$

$$\mathbf{H} := \mathbf{F} - \mathbf{G}. \tag{22.18c}$$

Denote the unknown coefficients of u as $\{\mathbf{U}_j\}_{j=0}^{n_0 - 1}$, i.e.,

$$u = \sum_{j=0}^{n_0 - 1} \mathbf{U}_j N^j \tag{22.19}$$

Finally, let J be the set of nodes on the Dirichlet boundary Γ_D, and $\overline{J} := \{0, ..., n_0 - 1\} \setminus J$. We get the system

$$\sum_{j=0}^{n_0 - 1} \mathbf{A}_{i,j} \mathbf{U}_j = \mathbf{H}_i, \qquad i \in \overline{J}, \tag{22.20a}$$

$$\mathbf{U}_i = g_D(N_i),\ i \in J. \tag{22.20b}$$

This leads to the system of equations

$$\sum_{j \in \overline{J}} \mathbf{A}_{i,j} x_j = \mathbf{H}_i - \sum_{j \in J} \mathbf{A}_{i,j} g_D(N_j),\ i \in \overline{J}. \tag{22.21}$$

Denote by $\mathbf{A}$ the restriction of A to the rows and columns in $\overline{J}$, by $\overline{\mathbf{H}}$ the right-hand side of the above equation (again only for the indices in $\overline{J}$), and by $\overline{\mathbf{U}}$ the restriction of $\mathbf{U}$ on the indices in $\overline{J}$. We arrive at the final linear system with positive-definite matrix $\overline{\mathbf{A}}$:

$$\overline{\mathbf{A}} \overline{\mathbf{U}} = \overline{\mathbf{H}}. \tag{22.22}$$

After solving it, we get the final solution

$$\overline{\mathbf{U}}_i = \begin{cases} \overline{\mathbf{U}}_i, & i \in \overline{J} \\ g_D(N_i), & i \in J \end{cases}. \tag{22.23}$$

22.2.2 Transient

Formulation 22.9. [Primal weak formulation for the transient discrete heat equation with discrete differential forms] The following formulation is a discrete version of Formulation 21.6. Let:

- Let d be a positive integer (space dimension);
- K be an oriented quasi-cubical mesh of dimension d representing the material body;
- $[K]$ be the fundamental class of K;

- $t_0 \in \mathbb{R}$ be the initial time;
- $I = [t_0, \infty)$;
- $f[ET^{-1}]\colon I \to C^d K$ be the internal production rate;
- $u_0[\Theta] \in C^0 K$ be the initial temperature;
- $\tilde{\pi}[EL^{-D}\Theta^{-1}]\colon I \times C^0 K \to C^0 K$ be the heat capacity of the material;
- $\tilde{\kappa}[EL^{2-D}T^{-1}\Theta^{-1}]\colon I \times C^1 K \to C^1 K$ be the thermal conductivity of the material, such that at any moment $t \in I$ and for any edge $c_1 \in K_1$, there exists some $\lambda > 0$ such that $\tilde{\kappa}(c^1) = \lambda c^1$;
- $\partial K = \Gamma_D \cup \Gamma_N$ be the partition of the boundary of K into Dirichlet (Γ_D) and Neumann (Γ_N) regions;
- $[\Gamma_N]$ be the fundamental class of Γ_N, where Γ_N has the boundary orientation induced from K;
- $g_D[\Theta]\colon I$ be the prescribed temperature on the Dirichlet boundary;
- $g_N[ET^{-1}]\colon I \to C^{D-1}\Gamma_N$ be the prescribed flow rate on the Neumann boundary.

define the following operators:

$$\begin{aligned}
&A\colon C^0K \times (I \to C^0K) \to \mathbb{R}, && A(v,w) := \langle \delta_0 v, \tilde{\kappa}\delta_0 w\rangle_{K,0} && [ET^{-1}\Theta^{-1}], && (22.24a)\\
&B\colon C^0K \times (I \to C^0K) \to \mathbb{R}, && B(v,w) := \langle v, \tilde{\pi} w\rangle_{K,0} && [E\Theta^{-1}], && (22.24b)\\
&G\colon C^0K \to \mathbb{R}, && G(v) := (\operatorname{tr}_{\Gamma_N} v \smile g_N)[\Gamma_N] && [ET^{-1}], && (22.24c)\\
&F\colon C^0K \to \mathbb{R}, && F(v) := (v \smile f)[K] && [ET^{-1}]. && (22.24d)
\end{aligned}$$

Our unknowns is temperature $u[\Theta]\colon I \to C^0 K$. We are solving the following problem for u:

$$\begin{aligned}
&\forall v[\Theta] \in \operatorname{Ker}\operatorname{tr}_{\Gamma_D,0}, \quad B(v, \frac{\partial u}{\partial t}) + A(v,u) = F(v) - G(v) && [ET^{-1}\Theta], && (22.25a)\\
&\operatorname{tr}_{\Gamma_D,0} u = g_D && [\Theta], && (22.25b)\\
&u(t_0) = u_0 && [\Theta]. && (22.25c)
\end{aligned}$$

The flow rate $q[ET^{-1}]\colon I \to C^{D-1}K$ is calculated in the post-processing phase by the formula

$$q(t, c_{D-1}) = \begin{cases} (- \star_1 \tilde{\kappa}\delta_0 u)(t, c_{D-1}), & c_{D-1} \in K_{D-1} \setminus (\Gamma_N)_{D-1} \\ g_N(t, c_{D-1}), & c_{D-1} \in (\Gamma_N)_{D-1} \end{cases}, \ t \in I. \tag{22.26}$$

Discussion 22.10. We are going to derive a solution to Formulation 22.9 using the trapezoidal rule for time integration. We will assume that the heat capacity $\tilde{\pi}$ is time-independent which will allow us to rearrange the time derivative:

$$B(v, \frac{\partial u}{\partial t}) = \frac{d}{dt} B(v,u). \tag{22.27}$$

For further simplicity we will also assume that all the rest input data internal production rate, thermal conductivity, boundary conditions) are also time-independent. Denote $H := F - G$. We can then integrate the equation

$$\frac{d}{dt} B(v,u) + A(v,u) = H(v) \tag{22.28}$$

with respect to t in the interval $[\alpha, \beta] \subset I$ to get

$$B(v, u(\beta)) - B(v, u(\alpha)) + A(v, \int_\alpha^\beta u \, dt) = (\beta - \alpha) H(v). \tag{22.29}$$

For an interval $[\alpha, \beta]$ the trapezoidal rule gives the approximation

$$A(v, \int_\alpha^\beta u \, dt) \approx A(v, \frac{\beta - \alpha}{2}(u(\alpha) + u(\beta))). \tag{22.30}$$

Hence, if we partition I into segements with size τ with moments of time $\{t_s := t_0 + \tau s\}_{s \geq 0}$, and if we denote $\{U^s := u(t_s)\}_{s \geq 0}$, we get

$$B(v, U^s) - B(v, U^{s-1}) + \frac{\tau}{2}(A(v, U^{s-1}) + A(v, U^s)) = \tau H(v). \tag{22.31}$$

The above equation is restated as

$$(B - \frac{\tau}{2}A)(v, U^s) = (B + \frac{\tau}{2}A)(v, U^{s-1}) + \tau H(v). \tag{22.32}$$

Define the left-hand side and right-hand side operators

$$L_\tau := B - \frac{\tau}{2}A, \tag{22.33a}$$

$$R_\tau := B + \frac{\tau}{2}A. \tag{22.33b}$$

The initial condition corresponds to $U^0 = u_0(t_0, \cdot)$. At any moment $s > 0$ we get the following problem for $U^s \in C^0 K$:

$$\forall v[\Theta] \in \operatorname{Ker} \operatorname{tr}_{\Gamma_D,0}, \quad L_\tau(v, U^s) = R_\tau(v, U^{s-1}) + \tau H(v), \tag{22.34a}$$

$$\operatorname{tr}_{\Gamma_D,0} U^s = g_D. \tag{22.34b}$$

As in the steady-state case Discussion 22.8, let J be the set of nodes on the Dirichlet boundary Γ_D, and $\overline{J} := \{0, ..., n_0 - 1\} \setminus J$. Denote the unknown coefficients of U^s as $\{\mathbf{U}_j^s\}_{j=0}^{n_0-1}$, i.e.,

$$U^s = \sum_{j=0}^{n_0-1} \mathbf{U}_j^s N^j. \tag{22.35}$$

In an analogous derivation to the one in Discussion 22.8, let $\overline{\mathbf{L}_\tau}$ be the matrix in the standard basis of the restriction of $\mathbf{L}_\tau$ to the rows and colums in $\overline{J}$, $\overline{\mathbf{U}}^s$ be the restriction of $\mathbf{U}^s$ on $\overline{J}$, and $\overline{\mathbf{H}_\tau} \in \mathbb{R}^{|\overline{J}|}$ be the vector defined by

$$\overline{\mathbf{H}_\tau} := \tau \mathbf{H}_i - \sum_{j \in J} (\mathbf{L}_\tau)_{i,j} g_D(N_j), \ i \in \overline{J}. \tag{22.36}$$

This leads to the system

$$\overline{\mathbf{L}_\tau}\, \overline{\mathbf{U}}^s = \overline{\mathbf{R}_\tau \mathbf{U}^{s-1}} + \overline{\mathbf{H}_\tau}, \tag{22.37}$$

where $\overline{\mathbf{R}_\tau \mathbf{U}^{s-1}}$ is the restriction of $\mathbf{R}_\tau \mathbf{U}^{s-1}$ to $\overline{J}$. This leads to the the following iterative process.

Algorithm 22.11 (Algorithm for solving the transient primal weak formulation for the discrete heat transfer phenomenon using trapezoidal rule for time integration, assuming time-independent input data)**.** Let:

- Let D be a positive integer (space dimension);
- K be an oriented quasi-cubical mesh of dimension D representing the material body;
- $[K]$ be the fundamental class of K;
- $t_0[T] \in \mathbb{R}$ be the initial time;
- $\tau[T] \in \mathbb{R}^+$ be the time step;
- $f[ET^{-1}] \in C^D K$ be the internal production rate;
- $u_0[\Theta] \in C^0 K$ be the initial temperature;
- $\tilde{\pi}[EL^{-D}\Theta^{-1}] \colon I \times C^0 K \to C^0 K$ be the heat capacity of the material;
- $\tilde{\kappa}[EL^{2-D}T^{-1}\Theta^{-1}] \colon C^1 K \to C^1 K$ be the thermal conductivity of the material;
- $\partial K = \Gamma_D \cup \Gamma_N$ be the partition of the boundary of K into Dirichlet (Γ_D) and Neumann (Γ_N) regions;
- $[\Gamma_N]$ be the fundamental class of Γ_N, where Γ_N has the boundary orientation induced from K;
- $g_D[\Theta] \in C^0 \Gamma_D$ be the prescribed temperature on the Dirichlet boundary;
- $g_N[ET^{-1}] \in C^{D-1} \Gamma_N$ be the prescribed flow rate on the Neumann boundary.

Our algorithm has 3 phases.

1. **Time-independent phase.** Do the following calculations:
   - $n_0 := |K_0|$;

- the sparse matrix $\mathbf{A} \in M_{n_0 \times n_0}(\mathbb{R})$,

$$\mathbf{A}_{i,j} := \langle \delta_0 N^j, \tilde{\kappa} \delta_0 N^i \rangle_{K,1}, \ i, j = 0, ..., n_0 - 1; \tag{22.38}$$

- the diagonal matrix $\mathbf{B} \in M_{n_0 \times n_0}(\mathbb{R})$,

$$\mathbf{B}_{i,j} := \langle N^j, \tilde{\pi} N^i \rangle_{K,0}, \ i, j = 0, ..., n_0 - 1; \tag{22.39}$$

- the right-hand side vectors $\mathbf{F}, \mathbf{G}, \mathbf{H} \in \mathbb{R}^{n_0}$,

$$\mathbf{F}_i := (N^i \smile f)[K], i = 0, ..., n_0 - 1, \tag{22.40a}$$
$$\mathbf{G}_i := (\mathrm{tr}_{\Gamma_N, 0} N^i \smile g_N)[\Gamma_N], i = 0, ..., n_0 - 1, \tag{22.40b}$$
$$\mathbf{H} := \mathbf{F} - \mathbf{G}; \tag{22.40c}$$

- the sparse matrices (having the same stencil as $\mathbf{A}$) $\mathbf{L}_\tau, \mathbf{R}_\tau \in M_{n_0 \times n_0}(\mathbb{R})$,

$$\mathbf{L}_\tau := \mathbf{B} - \frac{\tau}{2} \mathbf{A}, \tag{22.41a}$$
$$\mathbf{R}_\tau := \mathbf{B} + \frac{\tau}{2} \mathbf{A}; \tag{22.41b}$$

- the sets $J := (\Gamma_D)_0$ and $\overline{J} := \{0, ..., n_0 - 1\} \setminus J$;
- the vector $\widehat{\mathbf{U}} \in \mathbb{R}^{|J|}$ of the prescribed temperature on the Dirichlet boundary:

$$\widehat{\mathbf{U}}_i = g_D(N_i), \ i \in J; \tag{22.42}$$

- the restricted matrix $\overline{\mathbf{L}_\tau}$, constructed out of $\mathbf{L}_\tau$ with rows and columns in J removed, and the restricted vector $\overline{\mathbf{H}}_\tau \in \mathbb{R}^{|\overline{J}|}$

$$(\overline{\mathbf{H}}_\tau)_i := \tau \mathbf{H}_j - \sum_{j \in J} (\mathbf{L}_\tau)_{i,j} \widehat{\mathbf{U}}_j, \ i \in \overline{J}; \tag{22.43}$$

- the Cholesky decomposition

$$\overline{\mathbf{L}_\tau} = \overline{\mathbf{S}_\tau \mathbf{S}_\tau}^T, \tag{22.44}$$

where $\overline{\mathbf{S}_\tau}$ is a lower-triangular sparse matrix with positive diagonal;
- the time independent part of the restricted solution

$$\overline{\mathbf{Z}_\tau} := \overline{\mathbf{L}_\tau}^{-1} \overline{\mathbf{H}_\tau} = \overline{\mathbf{S}_\tau}^{-T} \overline{\mathbf{S}_\tau}^{-1} \overline{\mathbf{H}_\tau} \tag{22.45}$$

(of course, we do not find the inverses of $\overline{\mathbf{S}_\tau}$ and its transpose, but apply forward and back substitution);
- the initial coordinates $\mathbf{U}^0 \in \mathbb{R}^{n_0}$ of the temperature,

$$\mathbf{U}^0_i := u_0(N_i), \ i = 0, ..., n_0 - 1. \tag{22.46}$$

2. **Time-dependent (loop) phase.** For any $s > 0$ (until some predefined final moment is reached or some condition for small error is fulfilled) calculate:

- the time-dependent part $\overline{\mathbf{V}_\tau}^s$ of the right-hand side (allocated only once, updated on every step),

$$\overline{\mathbf{V}_\tau}^s := \overline{(\mathbf{R}_\tau \mathbf{U}^{s-1})}; \tag{22.47}$$

- the time-dependent part $\overline{\mathbf{W}_\tau}^s$ of the solution (allocated only once, updated on every step),

$$\overline{\mathbf{W}_\tau}^s := \overline{\mathbf{S}_\tau}^{-T} \overline{\mathbf{S}_\tau}^{-1} \overline{v_\tau} \tag{22.48}$$

(with forward and back substitution);
- the solution $\overline{\mathbf{U}}^s$ on the non-Dirichlet nodes (allocated only once, updated on every step),

$$\overline{\mathbf{U}}^s := \overline{\mathbf{W}_\tau}^s + \overline{\mathbf{Z}_\tau}; \tag{22.49}$$

- the final solution

$$\mathbf{U}^s_i := \begin{cases} \overline{\mathbf{U}}^s_i, & i \in \overline{J} \\ \widehat{\mathbf{U}}_i, & i \in J \end{cases}. \tag{22.50}$$

3. **Post-processing.** For each time moment t_s the flow rate $q^s \in C^{D-1} K$ is as follows: for any $c \in K_{D-1}$,

$$q^s(c_\bullet) := \begin{cases} (- \star_1 \circ \tilde{\kappa} \circ \delta_0 \, u^s)(c_\bullet), & c \in K_{D-1} \setminus (\Gamma_N)_{D-1} \\ g_N(c_\bullet), & c \in (\Gamma_N)_{D-1} \end{cases}. \tag{22.51}$$

22.3 Mixed weak formulation

22.3.1 Steady-state

Formulation 22.12. [Mixed weak formulation for the steady-state continuous heat equation with differential forms] The following formulation is a discrete version of Formulation 21.10. Let:

- D be a positive integer (space dimension);
- K be an oriented quasi-cubical mesh of dimension D representing the material body;
- $[K]$ be the fundamental class of K;
- $\kappa[EL^{2-D}T^{-1}\Theta^{-1}] \colon C^{D-1}K \to C^{D-1}K$ be the thermal conductivity of the material, such that for any hyperface $c_{D-1} \in K_{D-1}$, there exists some $\lambda > 0$ such that $\kappa(c^{D-1}) = \lambda c^{D-1}$;
- $f[ET^{-1}] \in C^D K$ be the internal production rate;
- $\partial K = \Gamma_D \cup \Gamma_N$ be the partition of the boundary of K into Dirichlet (Γ_D) and Neumann (Γ_N) regions;
- $[\Gamma_D]$ be the fundamental class of Γ_D, where Γ_D has the boundary orientation induced from K;
- $g_D[\Theta] \in C^0\Gamma_D$ be the prescribed temperature on the Dirichlet boundary;
- $g_N[ET^{-1}] \in C^{D-1}\Gamma_N$ be the prescribed flow rate on the Neumann boundary.

Define the following operators:

$$A \colon C^{D-1}K \times C^{D-1}K \to \mathbb{R}, \quad A(r, s) := \langle r, \kappa^{-1} s\rangle_{K,D-1} \quad [E^{-1}T\Theta], \tag{22.52a}$$
$$B \colon C^{D}K \times C^{D-1}K \to \mathbb{R}, \quad B(\tilde{v}, r) := \langle \delta_{D-1} r, \tilde{v}\rangle_{K,D} \quad [L^{-D}], \tag{22.52b}$$
$$G \colon C^{D-1}K \to \mathbb{R}, \quad G(r) := (\mathrm{tr}_{\Gamma_D, D-1}\, r \smile g_D)[\Gamma_D] \quad [\Theta], \tag{22.52c}$$
$$F \colon C^{D}K \to \mathbb{R}, \quad F(\tilde{v}) := \langle f, \tilde{v}\rangle_{K,D} \quad [ET^{-1}L^{-D}]. \tag{22.52d}$$

Our unknowns are:

- $q[ET^{-1}] \in C^{D-1}K$ (heat flow rate);
- $\tilde{u}[\Theta L^D] \in C^D K$ (dual temperature).

We are solving the following problem for q and $\tilde{u}$:

$$\forall r[ET^{-1}] \in \operatorname{Ker} \mathrm{tr}_{\Gamma_N, D-1}, \quad A(r, q) - B^T(r, \tilde{u}) = -G(r) \quad [ET^{-1}\Theta], \tag{22.53a}$$
$$\forall \tilde{v}[\Theta L^D] \in C^D K, \quad -B(\tilde{v}, q) = -F(\tilde{v}) \quad [ET^{-1}\Theta], \tag{22.53b}$$
$$\mathrm{tr}_{\Gamma_N, D-1}\, q = g_N \quad [ET^{-1}]. \tag{22.53c}$$

The temperature $u[\Theta] \in C^0 K$ is calculated in the post-processing phase by the formula

$$u(a_0) := \begin{cases} (\star_d \tilde{u})(a_0), & a_0 \notin (\Gamma_D)_0 \\ g_D(a_0), & a_0 \in (\Gamma_D)_0 \end{cases}. \tag{22.54}$$

Discussion 22.13. We are going to derive a solution to Formulation 22.12. For any $p \in \{0, ..., d\}$ denote

$$n_p := |K_p| = \dim(C_p K) = \dim(C^p K). \tag{22.55}$$

The cochains $(c^{p,0}, ..., c^{p,n_0 - 1})$ form the standard basis of $C^p K$. Define the diagonal matrix $\mathbf{A} \in M_{n_{D-1} \times n_{D-1}}(\mathbb{R})$, the sparse matrix $\mathbf{B} \in M_{n_d \times n_{D-1}}(\mathbb{R})$, and the vectors $\mathbf{F} \in \mathbb{R}^{n_d}$, $\mathbf{G} \in \mathbb{R}^{n_{D-1}}$ by

$$\mathbf{A}_{i,j} := \langle c^{d-1,j}, \kappa^{-1} c^{d-1,i}\rangle, \quad i, j = 0, ..., n_{D-1} - 1, \tag{22.56a}$$
$$\mathbf{B}_{k,i} := \langle \delta_{D-1} c^{d-1,i}, c^{d,k}\rangle, \quad k = 0, ..., n_d - 1, \ i = 0, ..., n_{D-1} - 1, \tag{22.56b}$$
$$\mathbf{F}_k := \langle f, c^{d,k}\rangle, \quad k = 0, ..., n_d - 1, \tag{22.56c}$$
$$\mathbf{G}_i := (\mathrm{tr}_{\Gamma_D, d-1}\, c^{d-1,i} \smile g_D)[\Gamma_D], \ i = 0, ..., n_{D-1} - 1. \tag{22.56d}$$

Denote the unknown coefficients of q as $\{\mathbf{Q}_j\}_{j=0}^{n_{D-1}-1}$, i.e.,

$$q = \sum_{j=0}^{n_{s-1}-1} \mathbf{Q}_j c^{d-1,j}, \tag{22.57}$$

and the unknown coefficients of $\tilde{u}$ as $\{\widetilde{\mathbf{U}}_k\}_{k=0}^{n_d-1}$, i.e.,

$$\tilde{u} = \sum_{k=0}^{n_d-1} \widetilde{\mathbf{U}}_k c^{d,k}. \tag{22.58}$$

Finally, let J be the set of $(d-1)$-cells on the Neumann boundary Γ_N, and $\overline{J} := \{0, ..., n_{D-1} - 1\} \setminus J$. We get the system

$$\begin{aligned}
\sum_{j=0}^{n_{D-1}-1} \mathbf{A}_{i,j}\mathbf{Q}_j - \sum_{k=0}^{n_d-1} (\mathbf{B}^T)_{i,k} y_k &= -\mathbf{G}_i, && i \in \overline{J}, && (22.59a)\\
-\sum_{i=0}^{n_{D-1}-1} \mathbf{B}_{k,i}\mathbf{Q}_i &= -\mathbf{F}_k, && k = 0, ..., n_d - 1, && (22.59b)\\
\mathbf{Q}_i &= g_N(c_{d-1,i}), && i \in J. && (22.59c)
\end{aligned}$$

This leads to the system of equations

$$\begin{aligned}
\sum_{j\in\overline{J}} \mathbf{A}_{i,j}\mathbf{Q}_j - \sum_{k=0}^{n_d-1} (\mathbf{B}^T)_{i,k} y_k &= -\mathbf{G}_i - \sum_{j\in J} \mathbf{A}_{i,j} g_N(c_{d-1,j}), \ i \in \overline{J}, && (22.60a)\\
-\sum_{i\in J} \mathbf{B}_{k,i}\mathbf{Q}_i &= -\mathbf{F}_k + \sum_{i\in J} \mathbf{B}_{k,i} g_N(c_{d-1,i}), \ k = 0, ..., n_d - 1. && (22.60b)
\end{aligned}$$

(Note that since $\mathbf{A}$ is diagonal, $\mathbf{A}_{i,j} = 0$ when $i \in \overline{J}$ and $j \in J$.)
Define the matrices $\overline{\mathbf{A}} \in M_{|\overline{J}|\times|\overline{J}|}(\mathbb{R})$, $\overline{\mathbf{B}} \in M_{n_d\times|\overline{J}|}(\mathbb{R})$, and vectors $\widehat{\mathbf{Q}} \in \mathbb{R}^{|J|}$, $\widetilde{\mathbf{F}} \in \mathbb{R}^{n_d}$, $\overline{\mathbf{G}} \in \mathbb{R}^{|\overline{J}|}$, $\overline{\mathbf{Q}} \in \mathbb{R}^{|\overline{J}|}$ by

$$\begin{aligned}
\overline{\mathbf{A}}_{i,j} &= \mathbf{A}_{i,j}, && i, j \in \overline{J}, && (22.61a)\\
\overline{\mathbf{B}}_{k,i} &= \mathbf{B}_{k,i}, && k = 0, ..., n_d - 1, \ i \in \overline{J}, && (22.61b)\\
\widehat{\mathbf{Q}}_i &= g_N(c_{d-1,i}), && i \in J, && (22.61c)\\
\widetilde{\mathbf{F}}_k &= \mathbf{F}_k - \sum_{i\in J} \mathbf{B}_{k,i}\widehat{\mathbf{Q}}_i, && k = 0, ..., n_d - 1, && (22.61d)\\
\overline{\mathbf{G}}_{k,i} &= \mathbf{G}_i + \sum_{j\in J} \mathbf{A}_{i,j}\widehat{\mathbf{Q}}_i, && i \in \overline{J}, && (22.61e)\\
\overline{\mathbf{Q}}_i &= \mathbf{Q}_i, && i \in \overline{J}. && (22.61f)
\end{aligned}$$

Hence, we get the folloeing system of equations for $\overline{\mathbf{Q}}$ and $\widetilde{\mathbf{U}}$:

$$\begin{aligned}
\overline{\mathbf{A}}\,\overline{\mathbf{Q}} - \overline{\mathbf{B}}^T\widetilde{\mathbf{U}} &= -\overline{\mathbf{G}}, && (22.62a)\\
-\overline{\mathbf{B}}\,\overline{\mathbf{Q}} &= -\widetilde{\mathbf{F}}. && (22.62b)
\end{aligned}$$

In general, when $\mathbf{A}$ is sparse but not diagonal, it is not beneficial to use the inverse of $\mathbf{A}$ in calculations since it will be a dense matrix. (This is the case in mixed finite element methods.) However, in our case $\mathbf{A}$ is diagonal, so the following calculation makes sense computationally. We can solve for $\overline{\mathbf{Q}}$ by

$$\overline{\mathbf{Q}} = \overline{\mathbf{A}}^{-1}(-\overline{\mathbf{G}} + \overline{\mathbf{B}}^T\widetilde{\mathbf{U}}). \tag{22.63}$$

Hence,

$$\widetilde{\mathbf{F}} = \overline{\mathbf{B}}\,\overline{\mathbf{Q}} = \overline{\mathbf{B}}\,\overline{\mathbf{A}}^{-1}(-\overline{\mathbf{G}} + \overline{\mathbf{B}}^T\widetilde{\mathbf{U}}). \tag{22.64}$$

This translates to

$$\overline{\mathbf{B}}\,\overline{\mathbf{A}}^{-1}\overline{\mathbf{B}}^T\widetilde{\mathbf{U}} = \overline{\mathbf{B}}\,\overline{\mathbf{A}}^{-1}\overline{\mathbf{G}} + \widetilde{\mathbf{F}}. \tag{22.65}$$

Using the Cholesky decomposition, we can solve for $\widetilde{\mathbf{U}}$. We then calculate $\overline{\mathbf{Q}}$ by substituting $\widetilde{\mathbf{U}}$ in Equation (22.63). Finally,

$$\mathbf{Q}_i = \begin{cases} \widehat{\mathbf{Q}}_i, & i \in J \\ \overline{\mathbf{Q}}_i, & i \in \overline{J} \end{cases}. \tag{22.66}$$

22.3.2 Transient

Formulation 22.14. [Mixed weak formulation for the discrete transient heat equation] The following formulation is a discrete version of Formulation 21.9. Let:

- d be a positive integer (space dimension);
- K be an oriented quasi-cubical mesh of dimension d representing the material body;
- $[K]$ be the fundamental class of K;
- $t_0 \in \mathbb{R}$ be the initial time;
- $I = [t_0, \infty)$ be the time-interval where the process occurs;
- $f[ET^{-1}] \colon I \to C^d K$ be the internal production rate;
- $u_0[\Theta] \in C^0 K$ be the initial temperature;
- $\kappa[EL^{2-D}T^{-1}\Theta^{-1}] \colon C^{D-1}K \to C^{D-1}K$ be the thermal conductivity of the material;
- $\pi[EL^{-D}\Theta^{-1}] \colon C^d K \to C^d K$ be the heat capacity of the material;
- $\partial K = \Gamma_D \cup \Gamma_N$ be the partition of the boundary of K into Dirichlet (Γ_D) and Neumann (Γ_N) regions;
- $[\Gamma_D]$ be the fundamental class of Γ_D, where Γ_D has the boundary orientation induced from K;
- $g_D[\Theta] \colon I \to C^0\Gamma_D$ be the prescribed temperature on the Dirichlet boundary;
- $g_N[ET^{-1}] \colon I \to C^{D-1}\Gamma_N$ be the prescribed flow rate on the Neumann boundary.

define the following operators:

$$
\begin{aligned}
&A \colon C^{D-1}K \times (I \to C^{D-1}K) \to \mathbb{R}, && A(r, s^{D-1}) := \langle r, \kappa^{-1} s^{D-1}\rangle_{K,d-1} && [E^{-1}T\Theta], && (22.67a)\\
&B \colon C^d K \times (I \to C^{D-1}K) \to \mathbb{R}, && B(v^d, r) := \langle \delta_{D-1} r, v^d\rangle_{K,d} && [L^{-D}], && (22.67b)\\
&C \colon C^d K \times (I \to C^d K) \to \mathbb{R}, && C(v^d, w^d) := \langle \pi w^d, v^d\rangle_{K,d} && [EL^{-2d}\Theta^{-1}], && (22.67c)\\
&G \colon C^{D-1}K \to \mathbb{R}, && G(r) := (\operatorname{tr}_{\Gamma_D, d-1} r \smile g_D)[\Gamma_D] && [\Theta], && (22.67d)\\
&F \colon C^d K \to \mathbb{R}, && F(v^d) := \langle f, v^d\rangle_{K,d} && [ET^{-1}L^{-D}]. && (22.67e)
\end{aligned}
$$

Our unknowns are:

- $q[ET^{-1}] \colon I \to C^{D-1}K$ (heat flow rate);
- $\tilde{u}[\Theta L^d] \colon I \to C^d K$ (dual temperature).

We are solving the following problem for q and $\tilde{u}$:

$$
\begin{aligned}
&\forall r[ET^{-1}] \in \operatorname{Ker} \operatorname{tr}_{\Gamma_N, d-1}, && A(r, q) - B^T(r, \tilde{u}) &&= -G(r) && [ET^{-1}\Theta], && (22.68a)\\
&\forall v^d[\Theta L^d] \in C^d K, && -B(v^d, q) - C(v^d, \frac{\partial \tilde{u}}{\partial t}) &&= -F(v^d) && [ET^{-1}\Theta], && (22.68b)\\
& && \operatorname{tr}_{\Gamma_N, d-1} q &&= g_N && [ET^{-1}], && (22.68c)\\
& && \tilde{u}(t_0) &&= \star_{K,0} u_0 && [\Theta L^d]. && (22.68d)
\end{aligned}
$$

The temperature $u[\Theta] \colon I \to C^0 K$ is calculated in the post-processing phase by the formula

$$
u(t, c_0) := \begin{cases} u_0(c_0), & t = t_0 \\ (\star_d \tilde{u})(t, x), & t > t_0 \text{ and } c_0 \notin (\Gamma_D)_0 \\ g_D(t, c_0), & t_0 > 0 \text{ and } c_0 \in (\Gamma_D)_0 \end{cases}. \tag{22.69}
$$

Discussion 22.15. We are going to derive a solution to Formulation 22.14 using the trapezoidal rule for time integration. We will assume that the heat capacity $\tilde{\pi}$ is time-independent which will allow us to rearrange the time derivative:

$$
C(\tilde{w}, \frac{\partial \tilde{u}}{\partial t}) = \frac{d}{dt} C(\tilde{w}, \tilde{u}). \tag{22.70}
$$

For further simplicity we will also assume that all the rest input data (heat source, thermal conductivity, boundary conditions) are also time-independent. We can then integrate the equation (the conservation law)

$$-B(\tilde{w}, q) - \frac{d}{dt}C(\tilde{w}, \tilde{u}) = -F(\tilde{w}) \tag{22.71}$$

with respect to t in the interval $[\alpha, \beta] \subset I$ to get

$$-B(\tilde{w}, \int_\alpha^\beta q\, dt) - (C(\tilde{w}, \tilde{u}(\beta)) - C(\tilde{w}, \tilde{u}(\alpha))) = -(\beta - \alpha)F(\tilde{w}). \tag{22.72}$$

If we partition I into segements with size τ with moments of time $\{t_s := t_0 + \tau s\}_{s \geq 0}$, and if we denote

$$q^s := q(t_s), \;\; s \geq 0, \tag{22.73a}$$
$$\tilde{u}^s := \tilde{u}(t_s), \;\; s \geq 0, \tag{22.73b}$$

we get

$$-\frac{\tau}{2}(B(\tilde{w}, q^s) + B(\tilde{w}, q^{s+1})) - (C(\tilde{w}, \tilde{u}^{s+1}) - C(\tilde{w}, \tilde{u}^s)) = -\tau F(\tilde{w}). \tag{22.74}$$

By multiplying the above equation with $2/\tau$ and rearranging we get:

$$-B(\tilde{w}, q^{s+1}) - \frac{2}{\tau}C(\tilde{w}, \tilde{u}^{s+1}) = -2F(\tilde{w}) + B(\tilde{w}, q^s) - \frac{2}{\tau}C(\tilde{w}, \tilde{u}^s). \tag{22.75}$$

At step 0 we calculate initial data as

$$q^0 = (-\kappa \star_1 \delta_0)(u_0), \tag{22.76a}$$
$$\tilde{u}^0 = \star_0 u_0. \tag{22.76b}$$

At any step $s > 0$ we get the following system for $(q^s, \tilde{u}^s) \in C^{D-1}K \times C^D K$:

$$\forall r[ET^{-1}] \in \operatorname{Ker} \operatorname{tr}_{\Gamma_N, D-1}, A(r, q^s) - B^T(r, \tilde{u}^s) = -G(r), \tag{22.77a}$$
$$\forall \tilde{w}[\Theta L^D] \in C^D K, \quad - B(\tilde{w}, q^s) - \frac{2}{\tau}C(\tilde{w}, \tilde{u}^s) = -2F(\tilde{w}) + B(\tilde{w}, q^{s-1}) - \frac{2}{\tau}C(\tilde{w}, \tilde{u}^{s-1}), \tag{22.77b}$$
$$\operatorname{tr}_{\Gamma_N, d-1} q^s = g_N. \tag{22.77c}$$

Let

$$J := \{i \in \{0, ..., n_{D-1}\} \mid c_{d-1,i} \in (\Gamma_N)_{D-1}\}, \tag{22.78a}$$
$$\overline{J} := \{0, ..., n_{D-1}\} \setminus J. \tag{22.78b}$$

Initial conditions give us Q^0 and U^0. Denote the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and vectors $\mathbf{Q}^s$, $\mathbf{U}^s$, $\mathbf{F}$, $\mathbf{G}$ of the corresponding operators in standard bases. Let $s > 0$. We get the following system for $\mathbf{Q}^s$ and $\mathbf{U}^s$:

$$\sum_{j=0}^{n_{D-1}-1} \mathbf{A}_{i,j}\mathbf{Q}_j^s - \sum_{k=0}^{n_d-1}(\mathbf{B}^T)_{i,k}\mathbf{U}_k^s = -\mathbf{G}_i, \quad i \in \overline{J}, \tag{22.79a}$$
$$-\sum_{i=0}^{n_{D-1}-1} \mathbf{B}_{k,i}\mathbf{Q}^s - \frac{2}{\tau}(\mathbf{C}\mathbf{U}^s)_k = -2\mathbf{F}_k + \mathbf{B}\mathbf{Q}^{s-1} - \frac{2}{\tau}\mathbf{C}\mathbf{U}_k^{s-1}, k \in \{0, ..., n_d - 1\}, \tag{22.79b}$$
$$\mathbf{Q}_i^s = g_N(c_{d-1,i}), \quad i \in J. \tag{22.79c}$$

The system can be rewritten as:

$$\sum_{j \in \overline{J}} \mathbf{A}_{i,j}\mathbf{Q}_j^s - \sum_{k=0}^{n_d-1}(\mathbf{B}^T)_{i,k}\mathbf{U}_k^s = -\mathbf{G}_i - \sum_{j \in J}\mathbf{A}_{i,j}g_N(c_{d-1,j}), \quad i \in \overline{J}, \tag{22.80a}$$
$$-\sum_{i \in J}\mathbf{B}_{k,i}\mathbf{Q}_i^s - \sum_{l=0}^{n_d-1}\frac{2}{\tau}\mathbf{C}_{k,l}\mathbf{U}_l^s = -2\mathbf{F}_k + \sum_{i \in J}\mathbf{B}_{k,i}g_N(c_{d-1,i}) + \mathbf{B}\mathbf{Q}^{s-1} - \frac{2}{\tau}\mathbf{C}\mathbf{U}_k^{s-1}, i \in \overline{J}. \tag{22.80b}$$

Let $\overline{\mathbf{A}}$ be the restriction of $\mathbf{A}$ to the rows and colums in $\overline{J}$, $\overline{\mathbf{B}}$ be the restristion of $\mathbf{B}$ to the colums in $\overline{J}$, $\overline{\mathbf{Q}}^s$ be the restriction of $\mathbf{Q}^s$ to the indices in $\overline{J}$, $\widetilde{\mathbf{F}} \in \mathbb{R}^{n_d}$ be defined as,

$$\widetilde{\mathbf{F}}_k := 2\mathbf{F}_k - \sum_{i \in J}\mathbf{B}_{k,i}g_N(c_{d-1,i}), \;\; k = 0, ..., n_d - 1, \tag{22.81}$$

$\overline{\mathbf{G}}$ be the restriction of $\mathbf{G}$ to the indices in $\overline{J}$ (since A is diagonal, $\mathbf{G}$ is not modified before restriction). Hence, we get the restricted system

$$\overline{\mathbf{A}}\,\overline{\mathbf{Q}}^s - \overline{\mathbf{B}}^T\mathbf{U}^s = -\overline{\mathbf{G}}, \tag{22.82a}$$

$$-\overline{\mathbf{B}}\,\overline{\mathbf{Q}}^s - \frac{2}{\tau}\mathbf{C}\mathbf{U}^s = -\widetilde{\mathbf{F}} + \mathbf{B}\mathbf{Q}^{s-1} - \frac{2}{\tau}\mathbf{C}\mathbf{U}^{s-1}. \tag{22.82b}$$

We can solve for $\overline{\mathbf{Q}}^s$ as follows:

$$\overline{\mathbf{Q}}^s = \overline{\mathbf{A}}^{-1}(-\overline{\mathbf{G}} + \overline{\mathbf{B}}^T\mathbf{U}^s) = -\overline{\mathbf{P}} + \overline{\mathbf{R}}\mathbf{U}^s, \tag{22.83}$$

where we have denoted

$$\overline{\mathbf{P}} := \overline{\mathbf{A}}^{-1}\overline{\mathbf{G}}, \tag{22.84}$$

$$\overline{\mathbf{R}} := \overline{\mathbf{A}}^{-1}\overline{\mathbf{B}}^T. \tag{22.85}$$

This means that

$$-\overline{\mathbf{B}}\,\overline{\mathbf{Q}}^s - \frac{2}{\tau}\mathbf{C}\mathbf{U}^s = -\overline{\mathbf{B}\mathbf{A}}^{-1}(-\overline{\mathbf{G}} + \overline{\mathbf{B}}^T\mathbf{U}^s) - \frac{2}{\tau}\mathbf{C}\mathbf{U}^s = -(\overline{\mathbf{B}\mathbf{A}}^{-1}\overline{\mathbf{B}}^T + \frac{2}{\tau}\mathbf{C})\mathbf{U}^s + \overline{\mathbf{B}\mathbf{A}}^{-1}\overline{\mathbf{G}}. \tag{22.86}$$

Define the left-hand side matrix $\mathbf{N}_\tau \in M_{n_d \times n_d}(\mathbb{R})$,

$$\mathbf{N}_\tau := \overline{\mathbf{B}}\,\overline{\mathbf{A}}^{-1}\overline{\mathbf{B}}^T + \frac{2}{\tau}\mathbf{C}. \tag{22.87}$$

Hence, the conversation law becomes

$$\mathbf{N}_\tau\mathbf{U}^s = \widetilde{\mathbf{F}} + \overline{\mathbf{B}\mathbf{A}}^{-1}\overline{\mathbf{G}} - \mathbf{B}\mathbf{Q}^{s-1} + \frac{2}{\tau}\mathbf{C}\mathbf{U}^{s-1}. \tag{22.88}$$

Define the constant right-hand side vector $\mathbf{Z} \in \mathbb{R}^{n_d}$,

$$\mathbf{Z} := \overline{\mathbf{B}\mathbf{A}}^{-1}\overline{\mathbf{G}} + \widetilde{\mathbf{F}} = \overline{\mathbf{B}\mathbf{P}} + \widetilde{\mathbf{F}}. \tag{22.89}$$

This leads to the following linear $n_d \times n_d$ system:

$$\mathbf{N}_\tau\mathbf{U}^s = \mathbf{Z} - \mathbf{B}\mathbf{Q}^{s-1} + \frac{2}{\tau}\mathbf{C}\mathbf{U}^{s-1}. \tag{22.90}$$

Define

$$\mathbf{V}_\tau := \mathbf{N}_\tau^{-1}\mathbf{Z}, \tag{22.91}$$

$$\mathbf{Y}_\tau^s := -\mathbf{B}\mathbf{Q}^{s-1} + \frac{2}{\tau}\mathbf{C}\mathbf{U}^{s-1}, \tag{22.92}$$

$$\mathbf{W}_\tau^s := \mathbf{N}_\tau^{-1}\mathbf{Y}_\tau^s. \tag{22.93}$$

To find $\mathbf{V}_\tau$ and $\mathbf{W}_\tau^s$ we first find the Cholesky decomposition of $\mathbf{N}_\tau$:

$$\mathbf{N}_\tau = \mathbf{L}_\tau\mathbf{L}_\tau^T. \tag{22.94}$$

Hence,

$$\mathbf{U}^s = \mathbf{N}_\tau^{-1}(\mathbf{Z} - \mathbf{B}\mathbf{Q}^s + \frac{2}{\tau}\mathbf{C}\mathbf{U}^s) = \mathbf{V}_\tau + \mathbf{W}_\tau^s. \tag{22.95}$$

Summarasing, we get the following algorithmic procedure.

Algorithm 22.16 (Algorithm for solving the transient mixed weak formulation for the discrete heat transfer phenomenon using trapezoidal rule for time integration, assuming time-independent input data)**.** Let:

- Let D be a positive integer (space dimension);
- K be an oriented quasi-cubical mesh of dimension D representing the material body;
- $[K]$ be the fundamental class of K;
- $t_0[T] \in \mathbb{R}$ be the initial time;

- $\tau[T] \in \mathbb{R}^+$ be the time step;
- $f[ET^{-1}] \in C^D K$ be the internal production rate;
- $u_0[\Theta] \in C^0 K$ be the initial temperature;
- $\pi[EL^{-D}\Theta^{-1}]\colon I \times C^D K \to C^D K$ be the heat capacity of the material;
- $\kappa[EL^{2-D}T^{-1}\Theta^{-1}]\colon C^{D-1}K \to C^{D-1}K$ be the thermal conductivity of the material;
- $\partial K = \Gamma_D \cup \Gamma_N$ be the partition of the boundary of K into Dirichlet (Γ_D) and Neumann (Γ_N) regions;
- $[\Gamma_D]$ be the fundamental class of Γ_D, where Γ_D has the boundary orientation induced from K;
- $g_D[\Theta] \in C^0\Gamma_D$ be the prescribed temperature on the Dirichlet boundary;
- $g_N[ET^{-1}] \in C^{D-1}\Gamma_N$ be the prescribed flow rate on the Neumann boundary.

Our algorithm has 3 phases.

1. **Time-independent phase.** Calculate:
   - $n_p := |K_p|$, $p = D-1$ and $p = D$;
   - the diagonal matrix $\mathbf{A} \in M_{n_{D-1}\times n_{D-1}}(\mathbb{R})$,
   $$\mathbf{A}_{i,j} := \langle c^{D-1,j}, \kappa^{-1} c^{D-1,i}\rangle_{K,D-1},\ i,j = 0,...,n_{D-1}-1; \tag{22.96}$$
   - the sparse matrix $\mathbf{B} \in M_{n_d\times n_{D-1}}(\mathbb{R})$,
   $$\mathbf{B}_{k,i} := \langle \delta_{D-1} c^{D-1,i}, c^{D,k}\rangle_{K,D},\ i = 0,...,n_{D-1}-1, k = 0,...,n_d-1; \tag{22.97}$$
   - the diagonal matrix $\mathbf{C} \in M_{n_d\times n_d}(\mathbb{R})$,
   $$\mathbf{C}_{k,l} := \langle \pi c^{D,l}, c^{D,k}\rangle_{K,D},\ k,l = 0,...,n_d-1; \tag{22.98}$$
   - the (sparse) vector $\mathbf{G} \in \mathbb{R}^{n_{D-1}}$,
   $$\mathbf{G}_i := (\mathrm{tr}_{\Gamma_D,D-1}\, c^{D-1,i} \smile g_D)[\Gamma_D],\ i = 0,...,n_{D-1}-1; \tag{22.99}$$
   - the vector $\mathbf{F} \in \mathbb{R}^{n_d}$,
   $$\mathbf{F}_k := \langle f, c^{D,k}\rangle_{K,D},\ k = 0,...,n_d-1; \tag{22.100}$$
   - the sets
   $$J := \{i \in \{0,...,n_{D-1}\} \mid c_{D-1,i} \in (\Gamma_N)_{D-1}\}, \tag{22.101a}$$
   $$\overline{J} := \{0,...,n_{D-1}\} \setminus J; \tag{22.101b}$$
   - the restricted diagonal matrix $\overline{\mathbf{A}} \in M_{|\overline{J}|\times|\overline{J}|}(\mathbb{R})$ constructed out of $\mathbf{A}$ with rows and columns in J removed;
   - the restricted sparse matrix $\overline{\mathbf{B}} \in M_{n_d\times|\overline{J}|}(\mathbb{R})$ constructed out of $\mathbf{B}$ with columns in J removed;
   - the modified and restricted vector $\overline{\mathbf{G}} \in \mathbb{R}^{|\overline{J}|}$,
   $$\overline{\mathbf{G}}_i := \mathbf{G}_i + \sum_{j\in J} \mathbf{A}_{i,j} g_N(c_{D-1,j}),\ i \in \overline{J} \tag{22.102}$$
   (in our case A is diagonal and so for all $i \in \overline{J}$ we get $\overline{\mathbf{G}}_i = \mathbf{G}_i$, i.e., no modification);
   - the modified vector $\widetilde{\mathbf{F}} \in \mathbb{R}^{n_d}$,
   $$\widetilde{\mathbf{F}}_k := 2\mathbf{F}_k - \sum_{i\in J} \mathbf{B}_{k,i} g_N(c_{D-1,i}),\ k = 0,...,n_d-1; \tag{22.103}$$
   - the left-hand side matrix $\mathbf{N}_\tau \in M_{n_d\times n_d}(\mathbb{R})$,
   $$\mathbf{N}_\tau := \overline{\mathbf{B}}\,\overline{\mathbf{A}}^{-1}\overline{\mathbf{B}}^T + \frac{2}{\tau}\mathbf{C}; \tag{22.104}$$

- the lower-triangular positive definite sparse matrix $\mathbf{L}_\tau \in M_{n_d \times n_d}(\mathbb{R})$ realising the Cholesky decomposition

$$\mathbf{N}_\tau = \mathbf{L}_\tau \mathbf{L}_\tau^T; \tag{22.105}$$

- the time-independent part of the heat flow rate $\overline{\mathbf{P}} \in \mathbb{R}^{|\overline{J}|}$,

$$\overline{\mathbf{P}} := \overline{\mathbf{A}}^{-1} \overline{\mathbf{G}}; \tag{22.106}$$

- the time-independent matrix multiplier $\overline{\mathbf{R}} \in M_{|\overline{J}| \times n_d}(\mathbb{R})$,

$$\overline{\mathbf{R}} := \overline{\mathbf{A}}^{-1} \overline{\mathbf{B}}^T; \tag{22.107}$$

- the constant right-hand side vector $\mathbf{Z} \in \mathbb{R}^{n_d}$,

$$\mathbf{Z} := \overline{\mathbf{B}}\overline{\mathbf{P}} + \widetilde{\mathbf{F}}; \tag{22.108}$$

- the time-independent part of the dual temperature $\mathbf{V}_\tau \in \mathbb{R}^{n_d}$,

$$\mathbf{V}_\tau := \mathbf{N}_\tau^{-1} \mathbf{Z} = \mathbf{L}_\tau^{-T} \mathbf{L}_\tau^{-1} \mathbf{Z} \tag{22.109}$$

  (calculated with forward and back substitution);
- the initial cooefficients $\mathbf{U}^0 \in \mathbb{R}^{n_d}$ of the dual temperature, and $\mathbf{Q}^0 \in \mathbb{R}^{n_{D-1}}$ of the heat flow rate:

$$\mathbf{Q}_i^0 := (\text{flow\_rate}(\mathbf{u}_0))_i, \; i = 0, ..., n_{D-1} - 1, \tag{22.110a}$$
$$\mathbf{U}_k^0 := (\star_0 \mathbf{u}_0)_k, \qquad k = 0, ..., n_d - 1. \tag{22.110b}$$

Allocate memory for the following vectors, to be modified at each step in the looping phase (superscript index s only shows their time relevance):

- time-dependent part of the right-hand side $\mathbf{Y}_\tau^s \in \mathbb{R}^{n_d}$;
- time-dependent part of the solution $\mathbf{W}_\tau^s \in \mathbb{R}^{n_d}$;
- the restricted coordinates $\overline{\mathbf{Q}}^{s+1} \in \mathbb{R}^{|\overline{J}|}$.

2. **Time-dependent (loop) phase.** The constant input consists of g_N, $\mathbf{B}$, $\overline{\mathbf{B}}$, $\mathbf{C}$, τ, $\mathbf{L}_\tau$, $\mathbf{V}_\tau$, $\overline{\mathbf{P}}$. Temporary mutable variables include the vectors $\mathbf{Y}_\tau^s$, $\mathbf{W}_\tau^s$, $\overline{\mathbf{Q}}^{s+1}$. The output consists of the coordinates $\mathbf{Q}$ of the heat flow rate, and the coordinates $\mathbf{U}$ of dual temperature. $\mathbf{Q}$ and $\mathbf{U}$ are either pre-allocated as arrays of size $(\texttt{number\_of\_time\_steps} + 1) \times n_p$ (for $p = D - 1$ and $p = D$ respectively) and initialized, or are only initialized, and memory is allocated at each step until some error condition is satisfied (e.g., the relative error between two consecutive steps becomes below some $\varepsilon > 0$, in which case the system converges to steady-state).

   For any $s > 0$ (until some predefined final moment is reached or some condition for small error is fulfilled), calculate:

   - RHS term $\mathbf{Y}_\tau^s$:

$$\mathbf{Y}_\tau^s := -\mathbf{B}\mathbf{Q}^{s-1} + \frac{2}{\tau}\mathbf{C}\mathbf{U}^{s-1}; \tag{22.111}$$

   - solution term $\mathbf{W}_\tau^s$:

$$\mathbf{W}_\tau^s := \mathbf{N}_\tau^{-1} \mathbf{Y}_\tau^s = \mathbf{L}_\tau^{-T} \mathbf{L}_\tau^{-1} \mathbf{Y}_\tau^s; \tag{22.112}$$

     (calculated with forward and back substitution);
   - the coefficients of the dual temperature $\mathbf{U}^s$ (stored for all s),

$$\mathbf{U}^s := \mathbf{V}_\tau + \mathbf{W}_\tau^s; \tag{22.113}$$

   - the non-Neumann coefficients of the heat flow rate $\mathbf{Q}^s$,

$$\overline{\mathbf{Q}}^s := -\overline{\mathbf{P}} + \overline{\mathbf{R}}\mathbf{U}^s; \tag{22.114}$$

   - the heat flow rate $\mathbf{Q}^s \in \mathbb{R}^{n_{D-1}}$ (stored for all s),

$$\mathbf{Q}_j^s := \begin{cases} \overline{\mathbf{Q}}_j^s, & j \in \overline{J} \\ g_N(c_{D-1,j}), & j \in J \end{cases}. \tag{22.115}$$

3. **Post-processing.** Define the sets

$$J := \{i \in \{0, ..., n_0 - 1\} \mid c_{0,i} \in (\Gamma_D)_0\}, \quad (22.116a)$$
$$\overline{J} := \{0, ..., n_0 - 1\} \setminus J. \quad (22.116b)$$

For each time moment t_s the coordinates of the temperature $\mathbf{u}^s \in \mathbb{R}^{n_0}$ in the standard basis are calculated by the formula

$$\mathbf{u}_i^s := \begin{cases} (\star_D \mathbf{U}^{D,s})_i, & i \in \overline{J} \\ g_D(c_{0,i}), & i \in J \end{cases}. \quad (22.117)$$

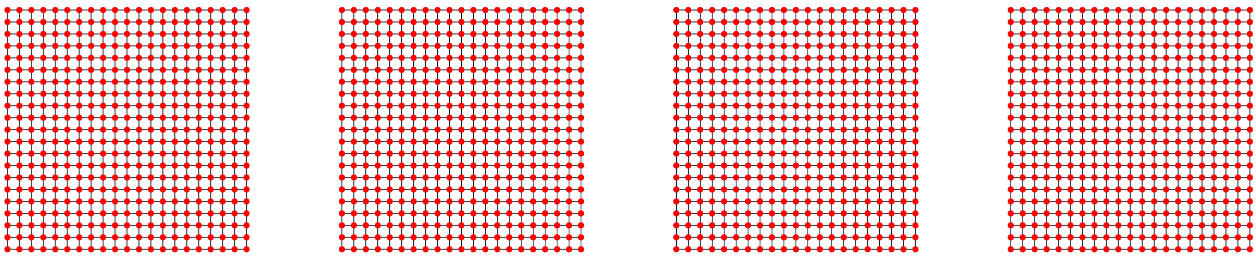
23 Examples of diffusion

23.1 Steady-state

Example 23.1. Consider the steady-state continuous heat transport problem (Formulation 21.4, Formulation 21.7, Formulation 21.10) with input data `2d_d00_p00` in the nomenclature of the C codebase.
Concretely, $X = [0, 1]^2$, $\tilde{\kappa} \equiv 1$, $f \equiv 0$, $G_D = \partial X$, $G_N = \emptyset$, $g_D(x, y) = 0$.
This problem has the following exact solution:

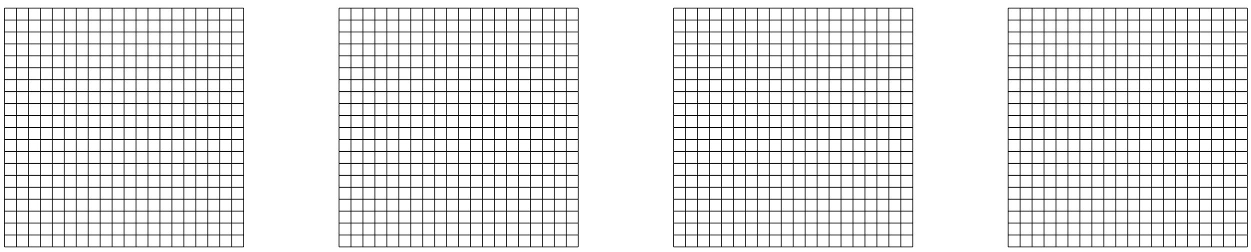
$$u \equiv 0, \quad (23.1a)$$
$$q \equiv 0. \quad (23.1b)$$

Consider a mesh M for X consisting of 10×10 squares (each axis is divided into 10 segments) with Forman subdivision K (20×20 squares). Its potential and flow rate on K consisting of the exact solution and the 3 discussed cochain methods are shown on Figure 2 and Figure 3.



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

Figure 2: `diffusion/steady_state/continuous_2d_d00_p00` (Example 23.1): solutions for potential



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

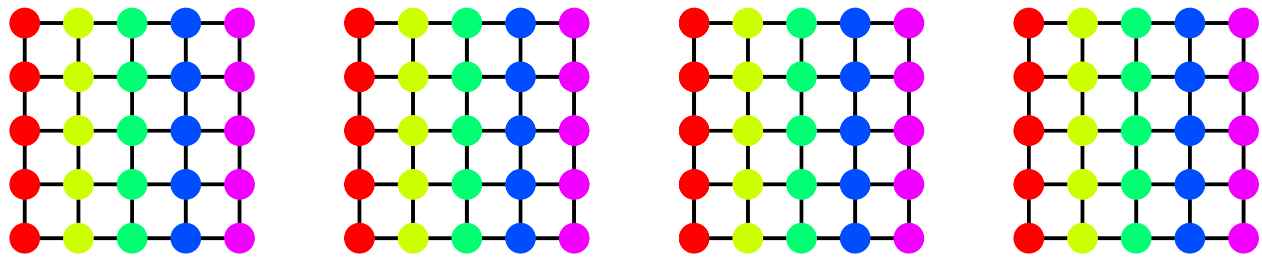
Figure 3: `diffusion/steady_state/continuous_2d_d00_p00` (Example 23.1): solutions for flow rate

Example 23.2. Consider the steady-state continuous heat transport problem (Formulation 21.4, Formulation 21.7, Formulation 21.10) with input data `2d_d00_p01` in the nomenclature of the C codebase.

Concretely, $X = [0,1]^2$, $\tilde{\kappa} \equiv 1$, $f \equiv 0$, $G_D = \{0,1\} \times [0,1]$, $G_N = [0,1] \times \{0,1\}$, $g_D(x,y) = \begin{cases} -100, & x = 0 \\ 100, & x = 1 \end{cases}$, $g_N \equiv 0$.
This problem has the following exact solution:

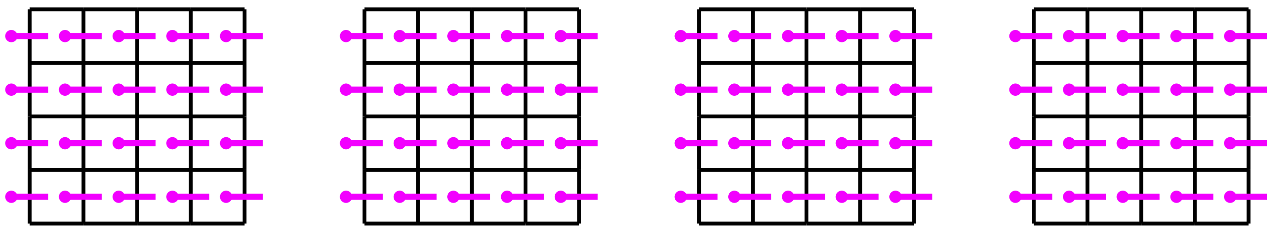
$$u(x,y) = 100(2x-1), \tag{23.2a}$$
$$q(x,y) = -200\,dy. \tag{23.2b}$$

Consider a mesh M for X consisting of 2×2 squares (each axis is divided into 2 segments) with Forman subdivision K (4×4 squares). Its potential and flow rate on K consisting of the exact solution and the 3 discussed cochain methods are shown on Figure 4 and Figure 5.



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

Figure 4: `diffusion/steady_state/continuous_2d_d00_p01` (Example 23.2): solutions for potential



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

Figure 5: `diffusion/steady_state/continuous_2d_d00_p01` (Example 23.2): solutions for flow rate

Example 23.3. Consider the steady-state continuous heat transport problem (Formulation 21.4, Formulation 21.7, Formulation 21.10) with input data `2d_d00_p02` in the nomenclature of the C codebase.
Concretely, $X = [0,1]^2$, $\tilde{\kappa} \equiv 1$, $f = -4\,dx \wedge dy$, $G_D = \partial X$, $G_N = \emptyset$, $g_D(x,y) = x^2 + y^2$.
This problem has the following exact solution:

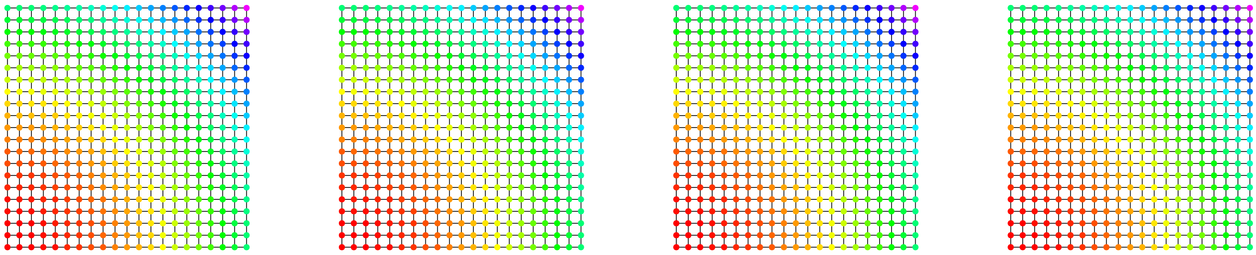
$$u(x,y) = x^2 + y^2, \tag{23.3a}$$
$$q(x,y) = 2y\,dx - 2x\,dy. \tag{23.3b}$$

Consider a mesh M for X consisting of 10×10 squares (each axis is divided into 10 segments) with Forman subdivision K (20×20 squares). Its potential and flow rate on K consisting of the exact solution and the 3 discussed cochain methods are shown on Figure 6 and Figure 7.

Example 23.4. Consider the steady-state continuous heat transport problem (Formulation 21.4, Formulation 21.7, Formulation 21.10) with input data `2d_d00_p03` in the nomenclature of the C codebase.
Concretely, $X = [0,1]^2$, $\tilde{\kappa} \equiv 1$, $f = -2\,dx \wedge dy$, $G_D = \{0,1\} \times [0,1]$, $G_N = [0,1] \times \{0,1\}$, $g_D \equiv 0$, $g_N \equiv 0$.
This problem has the following exact solution:

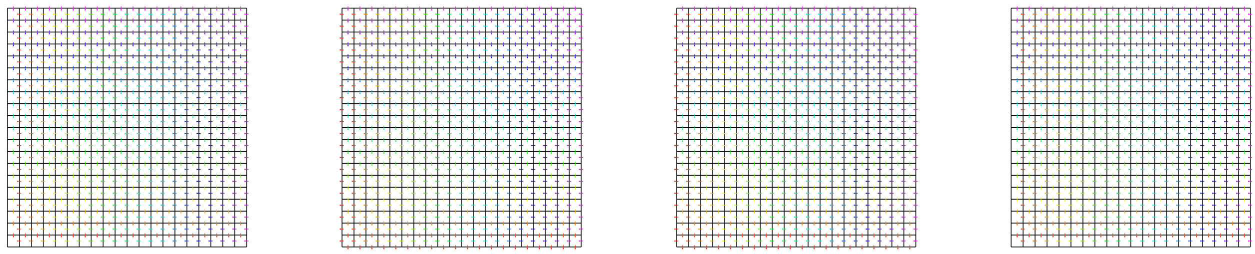
$$u(x,y) = x(x-1), \tag{23.4a}$$
$$q(x,y) = -(2x-1)\,dy. \tag{23.4b}$$

Consider a mesh M for X consisting of 2×2 squares (each axis is divided into 2 segments) with Forman subdivision



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

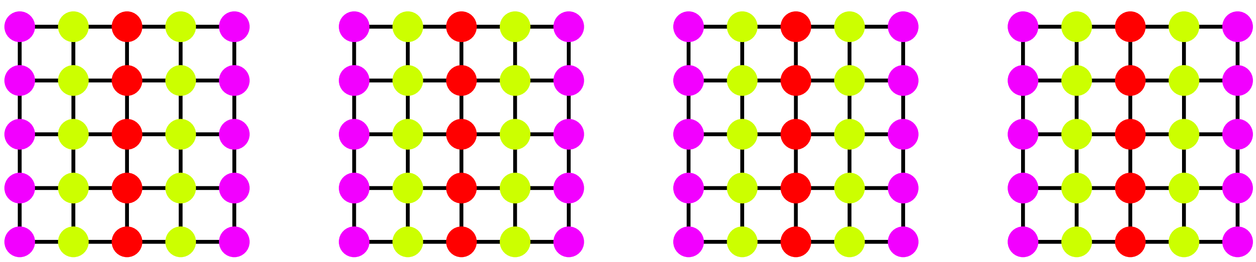
Figure 6: `diffusion/steady_state/continuous_2d_d00_p02` (Example 23.3): solutions for potential



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

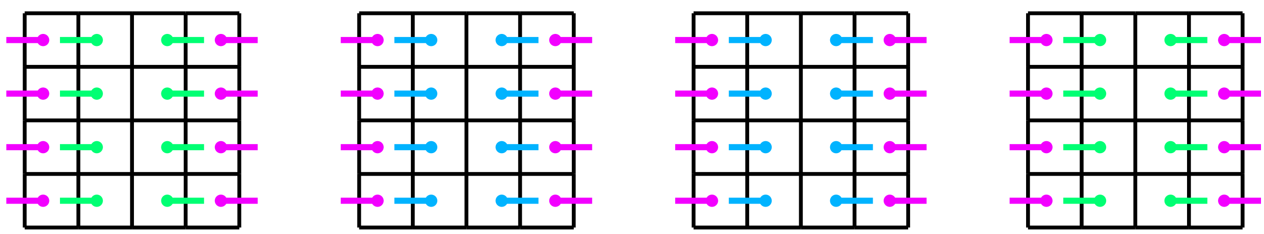
Figure 7: `diffusion/steady_state/continuous_2d_d00_p02` (Example 23.3): solutions for flow rate

K (4×4 squares). Its potential and flow rate on K consisting of the exact solution and the 3 discussed cochain methods are shown on Figure 8 and Figure 9.



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

Figure 8: `diffusion/steady_state/continuous_2d_d00_p03` (Example 23.4): solutions for potential



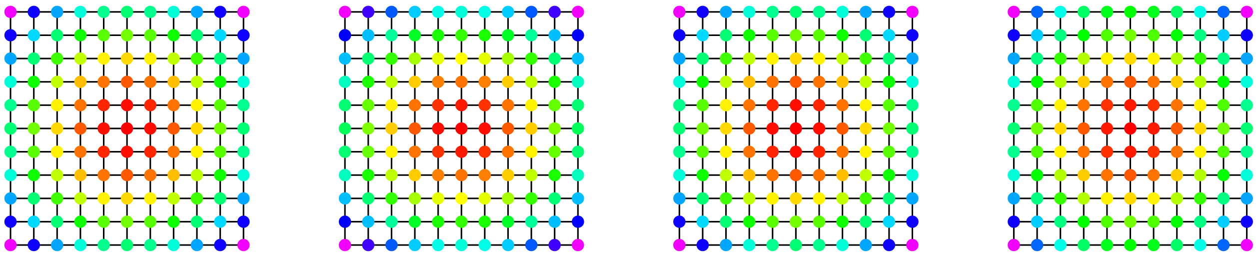
(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

Figure 9: `diffusion/steady_state/continuous_2d_d00_p03` (Example 23.4): solutions for flow rate

Example 23.5. Consider the steady-state continuous heat transport problem (Formulation 21.4, Formulation 21.7, Formulation 21.10) with input data `2d_d00_p04` in the nomenclature of the C codebase.
Concretely, $X = [0,1]^2$, $\tilde{\kappa} \equiv 1$, $f = -4\,dx \wedge dy$, $G_D = \{0,1\} \times [0,1]$, $G_N = [0,1] \times \{0,1\}$, $g_D(x,y) = y(y-1)$, $g_N(x,y) = (2y-1)\,dx = \begin{cases} -dx, & y = 0 \\ dx, & y = 1 \end{cases}$.
This problem has the following exact solution:

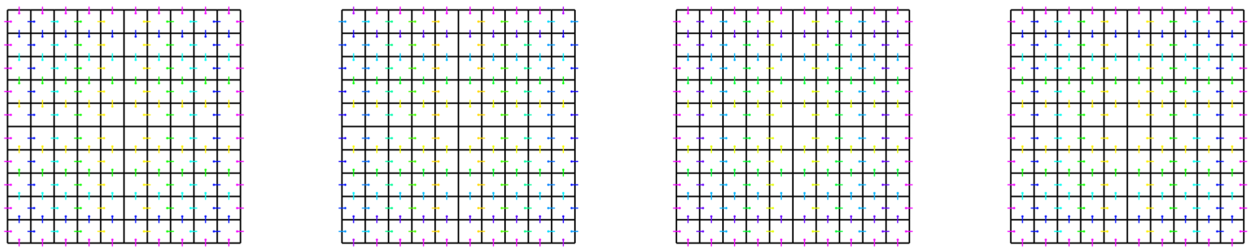
$$u(x,y) = x(x-1) + y(y-1), \tag{23.5a}$$
$$q(x,y) = (2y-1)\,dx - (2x-1)\,dy. \tag{23.5b}$$

Consider a mesh M for X consisting of 5×5 squares (each axis is divided into 5 segments) with Forman subdivision K (10×10 squares). Its potential and flow rate on K consisting of the exact solution and the 3 discussed cochain methods are shown on Figure 10 and Figure 11.



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

Figure 10: `diffusion/steady_state/continuous_2d_d00_p04` (Example 23.5): solutions for potential



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

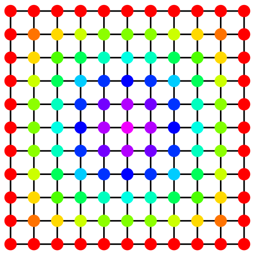
Figure 11: `diffusion/steady_state/continuous_2d_d00_p04` (Example 23.5): solutions for flow rate

Example 23.6. Consider the steady-state continuous heat transport problem (Formulation 21.4, Formulation 21.7, Formulation 21.10) with input data `2d_d00_p05` in the nomenclature of the C codebase.
Concretely, $X = [0,1]^2$, $\tilde{\kappa} \equiv 1$, $f(x,y) = \sin(\pi x)\sin(\pi y)\,dx \wedge dy$, $G_D = \partial X$, $G_N = \emptyset$, $g_D \equiv 0$.
This problem has the following exact solution:

$$u(x,y) = \frac{\sin(\pi x)\sin(\pi y)}{2\pi^2}, \tag{23.6a}$$
$$q(x,y) = \frac{1}{2\pi}((\sin(\pi x)\cos(\pi y)\,dx - \sin(\pi y)\cos(\pi x)\,dy)). \tag{23.6b}$$

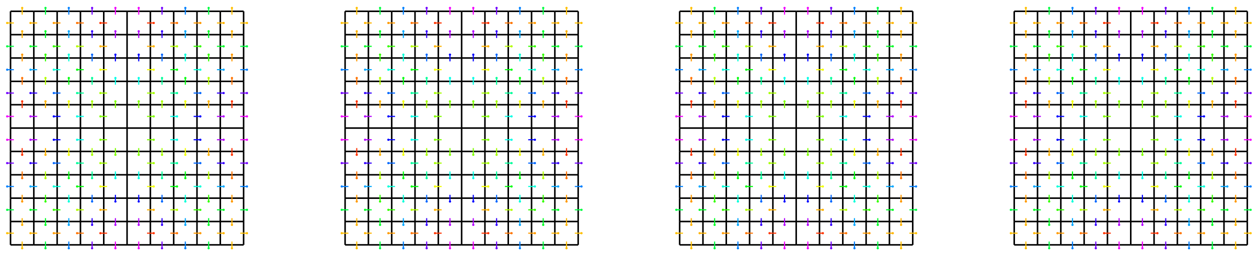
Consider a mesh M for X consisting of 5×5 squares (each axis is divided into 5 segments) with Forman subdivision K (10×10 squares). Its potential and flow rate on K consisting of the exact solution and the 3 discussed cochain methods are shown on Figure 12 and Figure 13.

Example 23.7. Consider the steady-state continuous heat transport problem (Formulation 21.4, Formulation 21.7, Formulation 21.10) with input data `2d_d01_p00` in the nomenclature of the C codebase.
Concretely, $X = \text{polygon}((-5,0),(0,-5),(5,0),(0,5)$, $\tilde{\kappa} \equiv 6$, $f \equiv 0$, $G_D = \text{line}((-5,0),(0,-5)) \cup \text{line}((5,0),(0,5))$,



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

Figure 12: `diffusion/steady_state/continuous_2d_d00_p05` (Example 23.6): solutions for potential



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

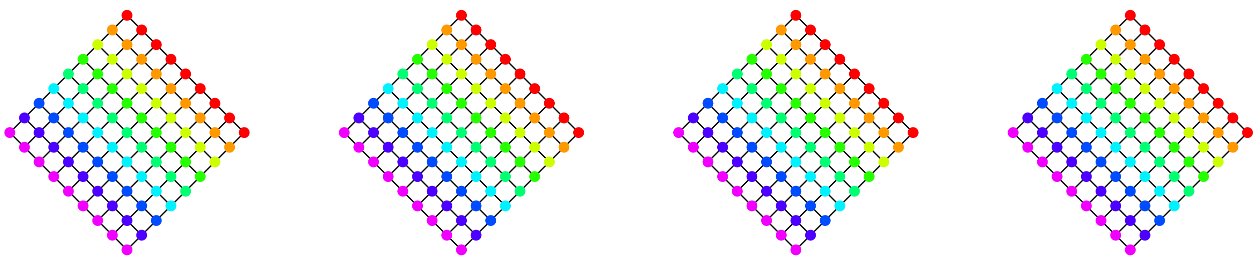
Figure 13: `diffusion/steady_state/continuous_2d_d00_p05` (Example 23.6): solutions for flow rate

$G_N = \mathrm{line}((0,-5),(5,0)) \cup \mathrm{line}((0,5),(-5,0))$, $g_D(x,y) = \begin{cases} 100, & (x,y) \in \mathrm{line}((-5,0),(0,-5)) \\ 0, & (x,y) \in \mathrm{line}((5,0),(0,5)) \end{cases}$, $g_N \equiv 0$.

This problem has the following exact solution:

$$u(x,y) = 50(1 - (x+y)/5), \tag{23.7a}$$
$$q(x,y) = 60(-dx + dy). \tag{23.7b}$$

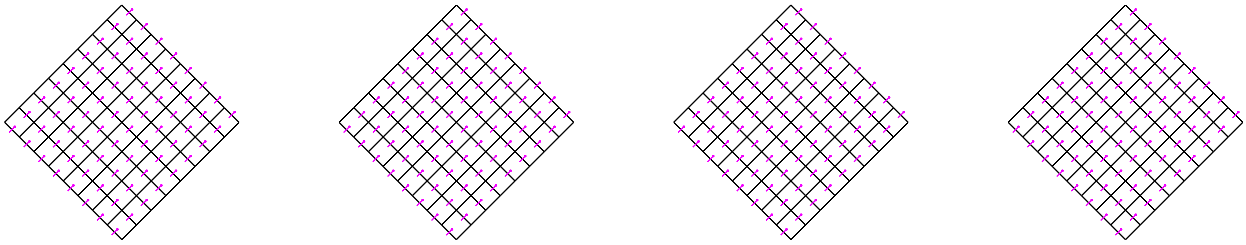
Consider a mesh M for X consisting of 4×4 squares (each axis is divided into 4 segments) with Forman subdivision K (8×8 squares). Its potential and flow rate on K consisting of the exact solution and the 3 discussed cochain methods are shown on Figure 14 and Figure 15.



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

Figure 14: `diffusion/steady_state/continuous_2d_d01_p00` (Example 23.7): solutions for potential

Example 23.8. Consider the steady-state continuous heat transport problem (Formulation 21.4, Formulation 21.7, Formulation 21.10) with input data `2d_d02_p00` in the nomenclature of the C codebase.
Concretely, $X = [0,20] \times [0,15]$, $\tilde{\kappa} \equiv 6$, $f \equiv 0$, $G_D = \{0,20\} \times [0,15]$, $G_N = [0,20] \times \{0,15\}$, $g_D(x,y) = \begin{cases} 100, & x = 0 \\ 0, & x = 20 \end{cases}$, $g_N \equiv 0$.



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

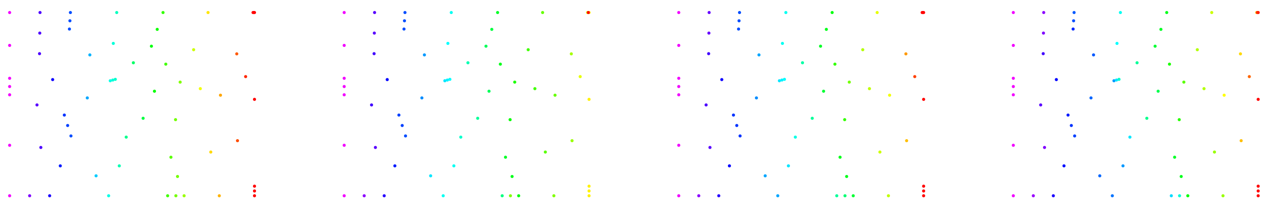
Figure 15: `diffusion/steady_state/continuous_2d_d01_p00` (Example 23.7): solutions for flow rate

This problem has the following exact solution:

$$u(x, y) = 5(20 - x), \tag{23.8a}$$

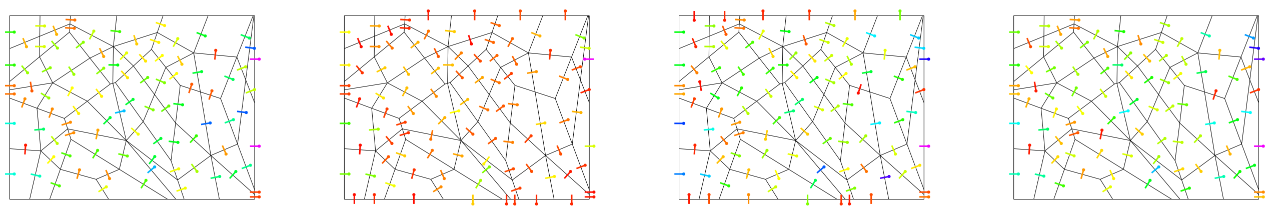
$$q(x, y) = 30\, dy. \tag{23.8b}$$

For this problem I use a mesh M generated by Neper with Forman subdivision K. Its potential and flow rate on K consisting of the exact solution and the 3 discussed cochain methods are shown on Figure 16 and Figure 17.



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

Figure 16: `diffusion/steady_state/continuous_2d_d02_p00` (Example 23.8): solutions for potential



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

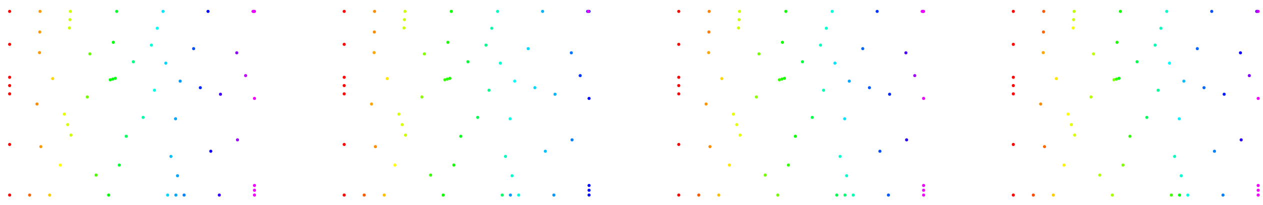
Figure 17: `diffusion/steady_state/continuous_2d_d02_p00` (Example 23.8): solutions for flow rate

Example 23.9. Consider the steady-state continuous heat transport problem (Formulation 21.4, Formulation 21.7, Formulation 21.10) with input data `2d_d02_p01` in the nomenclature of the C codebase.
Concretely, $X = [0, 20] \times [0, 15]$, $\tilde{\kappa} \equiv 6$, $f \equiv 0$, $G_D = \{0, 20\} \times [0, 15]$, $G_N = [0, 20] \times \{0, 15\}$, $g_D(x, y) = \begin{cases} 0, & x = 0 \\ 100, & x = 20 \end{cases}$, $g_N \equiv 0$.
This problem has the following exact solution:

$$u(x, y) = 5x, \tag{23.9a}$$

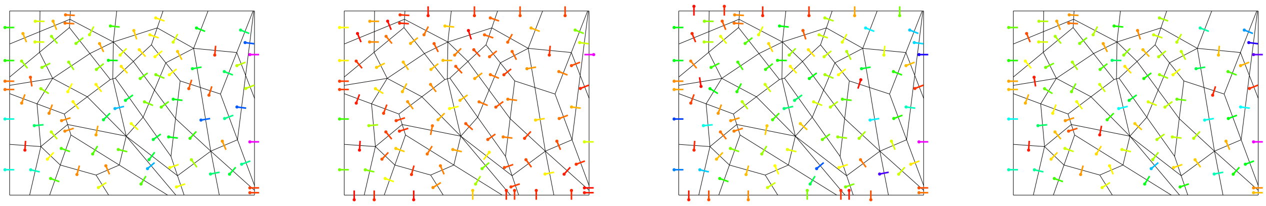
$$q(x, y) = -30\, dy. \tag{23.9b}$$

For this problem I use a mesh M generated by Neper with Forman subdivision K. Its potential and flow rate on K consisting of the exact solution and the 3 discussed cochain methods are shown on Figure 18 and Figure 19.



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

Figure 18: `diffusion/steady_state/continuous_2d_d02_p01` (Example 23.9): solutions for potential



(a) Exact (b) Primal strong (c) Primal weak (d) Mixed weak

Figure 19: `diffusion/steady_state/continuous_2d_d02_p01` (Example 23.9): solutions for flow rate

Example 23.10. Consider the steady-state continuous heat transport problem (Formulation 21.4, Formulation 21.7, Formulation 21.10) with input data `2d_d03_p00` in the nomenclature of the C codebase.
Concretely, $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, $\tilde{\kappa} \equiv 1$, $f = -4\, dx \wedge dy$, $G_D = \partial X$, $G_N = \emptyset$, $g_D \equiv 1$.
This problem has the following exact solution:

$$u(x, y) = x^2 + y^2, \tag{23.10a}$$
$$q(x, y) = 2y\, dx - 2x\, dy. \tag{23.10b}$$

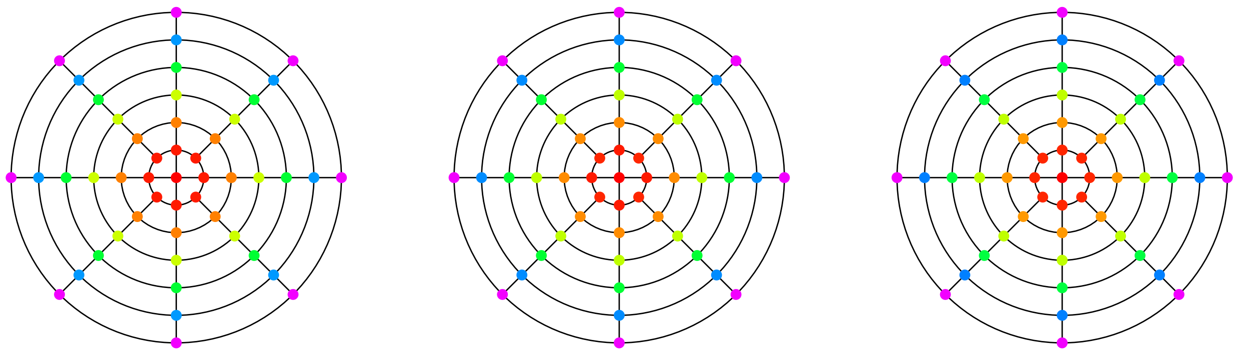
Consider a mesh M for X consisting of n_a rays and n_d disks with Forman subdivision K. Its potential and flow rate on K consisting of the exact solution and the 2 of the discussed cochain methods (no primal strong) are shown on Figure 20, Figure 21 ($(n_a, n_d) = (4, 3)$) and Figure 22, Figure 23 ($(n_a, n_d) = (18, 10)$).

Example 23.11. Consider the steady-state continuous heat transport problem (Formulation 21.4, Formulation 21.7, Formulation 21.10) with input data `2d_d03_p01` in the nomenclature of the C codebase.
Concretely, $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, $\tilde{\kappa} \equiv 1$, $f = -4\, dx \wedge dy$, $G_D = \{(x, y) \in \partial X \mid x \geq 0\}$, $G_N = \{(x, y) \in \partial X \mid x \leq 0\}$, $g_D \equiv 1$, $g_N(t) = -2\, dt$ (with respect to the $(x, y) = (\cos(t), \sin(t))$ coordinates).
This problem has the following exact solution:

$$u(x, y) = x^2 + y^2, \tag{23.11a}$$
$$q(x, y) = 2y\, dx - 2x\, dy. \tag{23.11b}$$

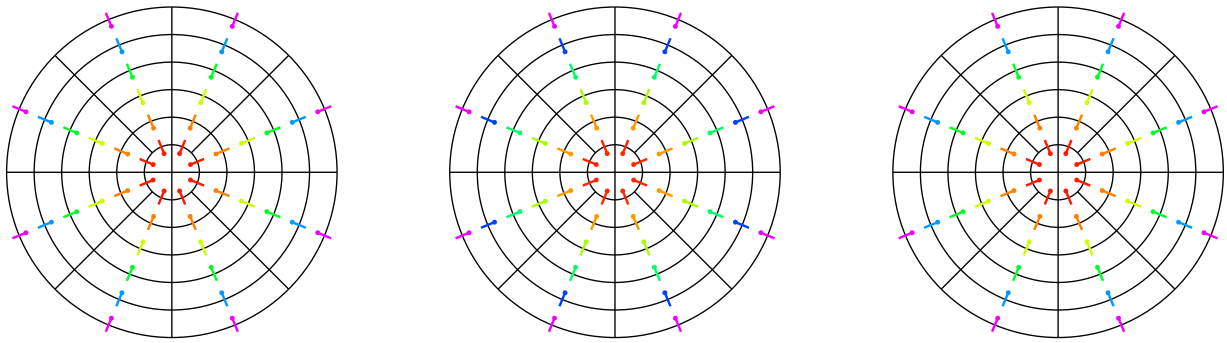
Consider a mesh M for X consisting of n_a rays and n_d disks (n_a must be even) with Forman subdivision K. Its potential and flow rate on K consisting of the exact solution and the 2 of the discussed cochain methods (no primal strong) are shown on Figure 24, Figure 25 ($(n_a, n_d) = (4, 3)$) and Figure 26, Figure 27 ($(n_a, n_d) = (18, 10)$).

Example 23.12. Consider the steady-state continuous heat transport problem (Formulation 21.4, Formulation 21.7, Formulation 21.10) with input data `2d_d04_p00` in the nomenclature of the C codebase.



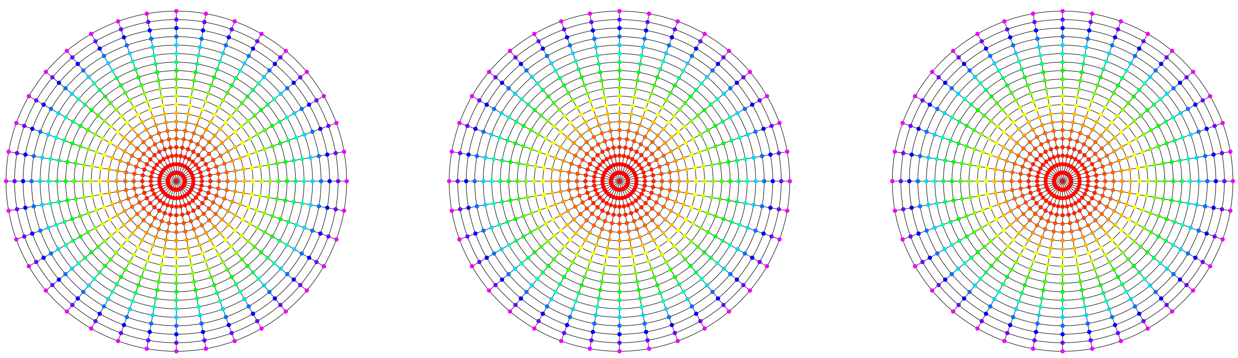
(a) Exact (b) Primal weak (c) Mixed weak

Figure 20: `diffusion/steady_state/continuous_2d_d03_p00` (Example 23.10): solutions for potential on mesh `disk_polar_4_3_forman`



(a) Exact (b) Primal weak (c) Mixed weak

Figure 21: `diffusion/steady_state/continuous_2d_d03_p00` (Example 23.10): solutions for flow rate on mesh `disk_polar_4_3_forman`

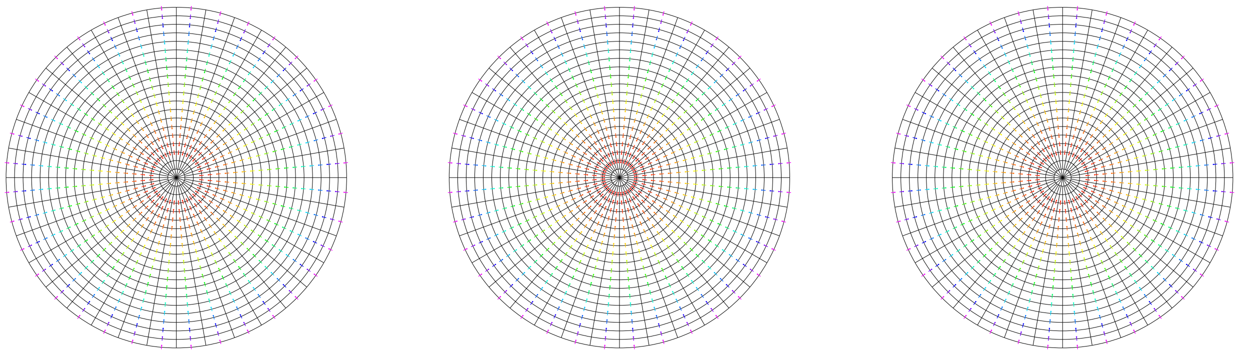


(a) Exact (b) Primal weak (c) Mixed weak

Figure 22: `diffusion/steady_state/continuous_2d_d03_p00` (Example 23.10): solutions for potential on mesh `disk_polar_18_10_forman`

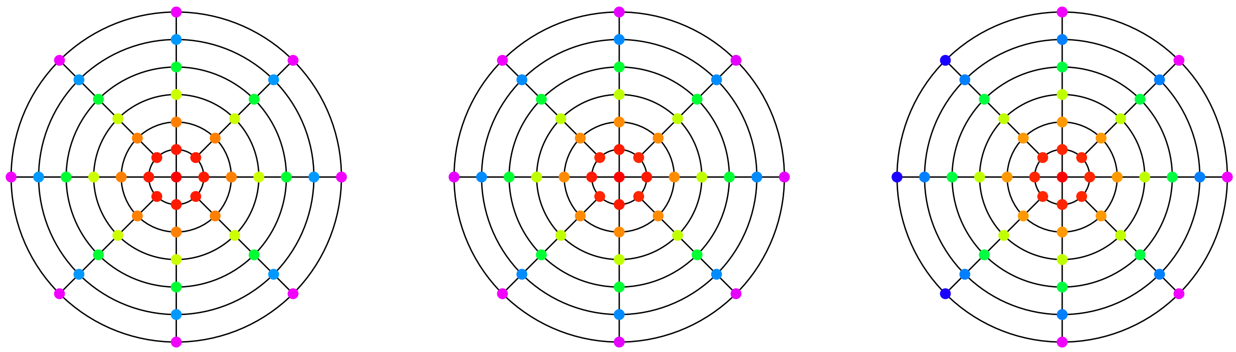
Concretely, $X = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ z \geq 0\right\}$ be a hemisphere with the induced metric, $\tilde{\kappa} \equiv 2$, $f = 6\kappa(x^2 - y^2)\,\mathrm{vol}$, $G_D = \partial X$, $G_N = \emptyset$, $g_D(x, y, z) = x^2 - y^2$.
Use spherical coordinates

$$(x, y, z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),\ 0 \leq \theta \leq \pi/2,\ 0 \leq \varphi \leq 2\pi. \tag{23.12}$$



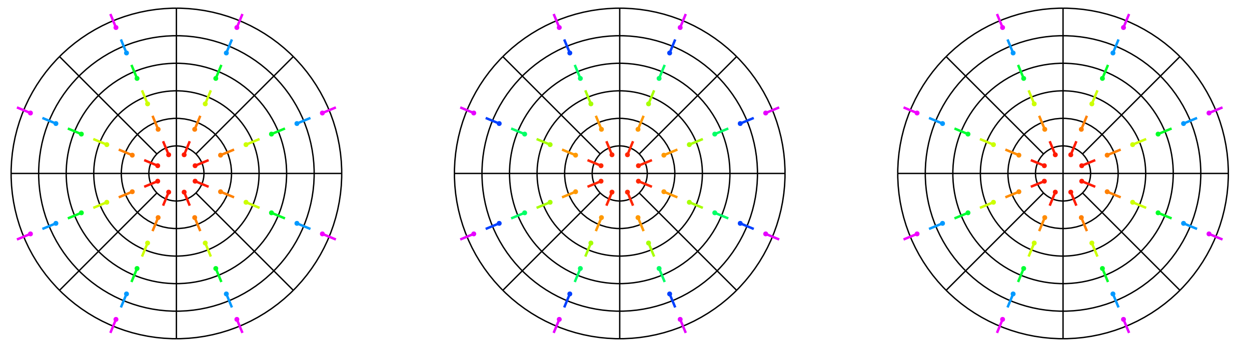
(a) Exact (b) Primal weak (c) Mixed weak

Figure 23: `diffusion/steady_state/continuous_2d_d03_p00` (Example 23.10): solutions for flow rate on mesh `disk_polar_18_10_forman`



(a) Exact (b) Primal weak (c) Mixed weak

Figure 24: `diffusion/steady_state/continuous_2d_d03_p01` (Example 23.11): solutions for potential on mesh `disk_polar_4_3_forman`



(a) Exact (b) Primal weak (c) Mixed weak

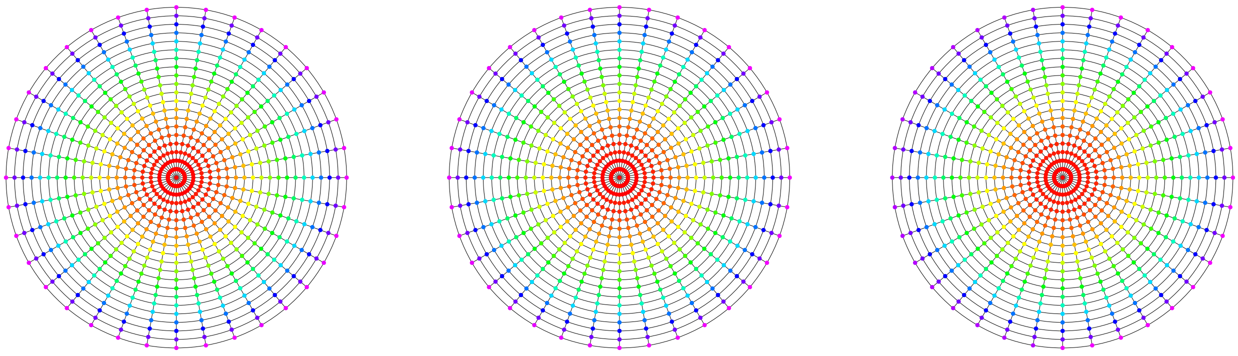
Figure 25: `diffusion/steady_state/continuous_2d_d03_p01` (Example 23.11): solutions for flow rate on mesh `disk_polar_4_3_forman`

This problem has the following exact solution:

$$u(x,y,z) = x^2 - y^2, \tag{23.13a}$$

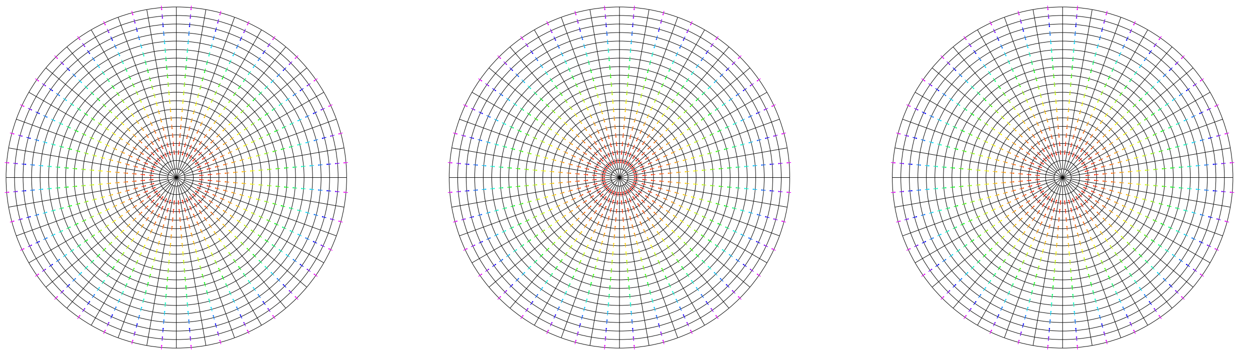
$$\tilde{q}(\theta,\varphi) = 2\kappa(-2\sin\theta\sin(2\varphi)\,d\theta - \sin\theta\sin(2\theta)\cos(2\varphi)\,d\varphi). \tag{23.13b}$$

Consider a mesh M for X consisting of n_a meridians and n_d parallels with Forman subdivision K. Its potential and



(a) Exact (b) Primal weak (c) Mixed weak

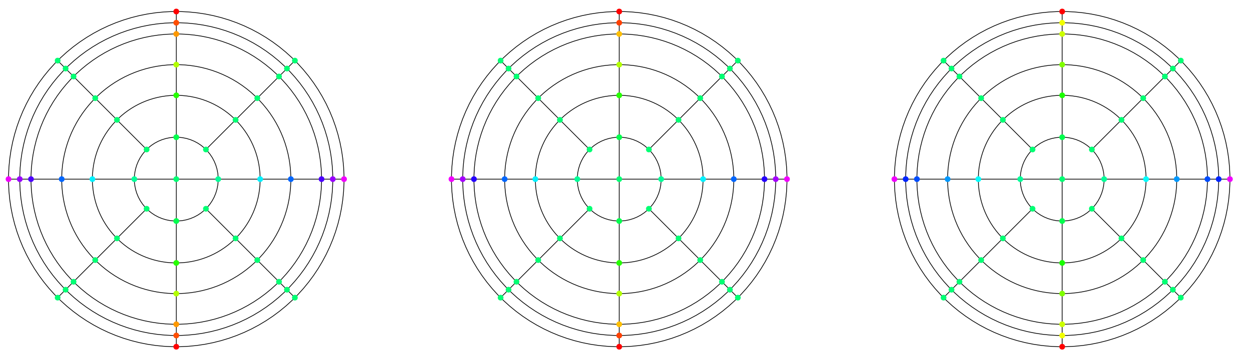
Figure 26: `diffusion/steady_state/continuous_2d_d03_p01` (Example 23.11): solutions for potential on mesh `disk_polar_18_10_forman`



(a) Exact (b) Primal weak (c) Mixed weak

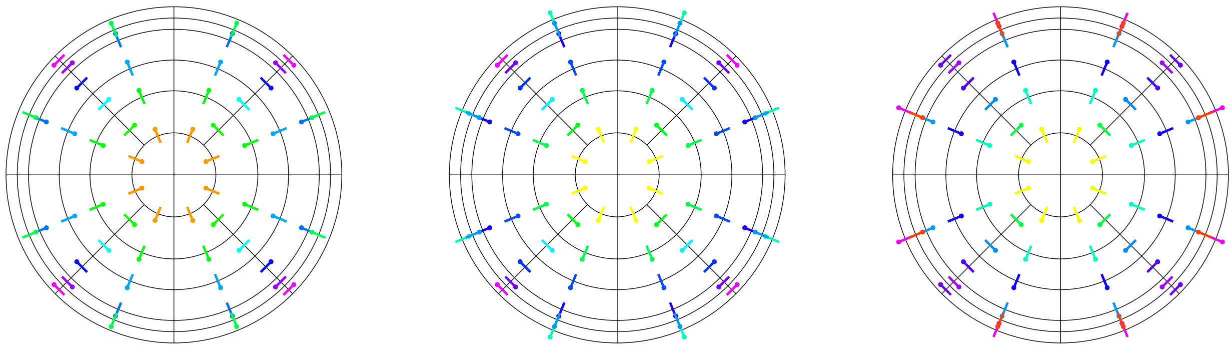
Figure 27: `diffusion/steady_state/continuous_2d_d03_p01` (Example 23.11): solutions for flow rate on mesh `disk_polar_18_10_forman`

flow rate on the xy-projection of K consisting of the exact solution and the 2 of the discussed cochain methods (no primal strong) are shown on Figure 28, Figure 29 ($(n_a, n_d) = (4, 3)$) and Figure 30, Figure 31 ($(n_a, n_d) = (6, 6)$).



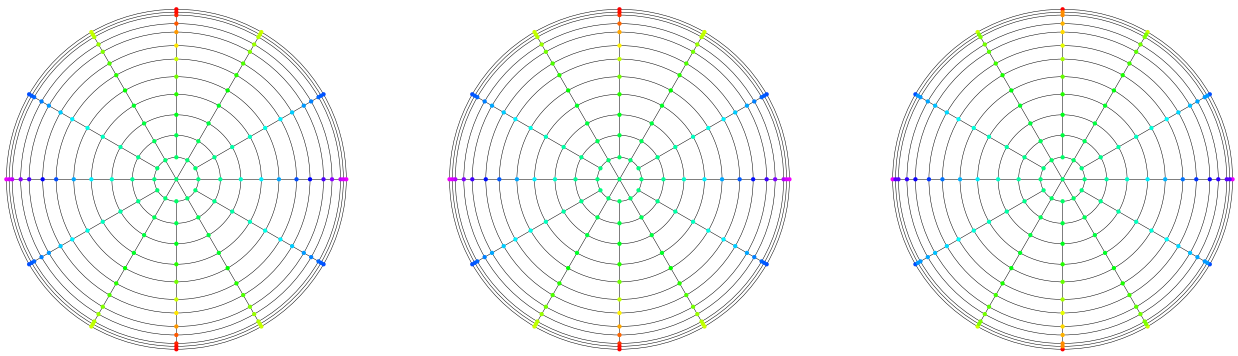
(a) Exact (b) Primal weak (c) Mixed weak

Figure 28: `diffusion/steady_state/continuous_2d_d04_p00` (Example 23.12): solutions for potential on mesh `hemisphere_polar_4_3_forman`



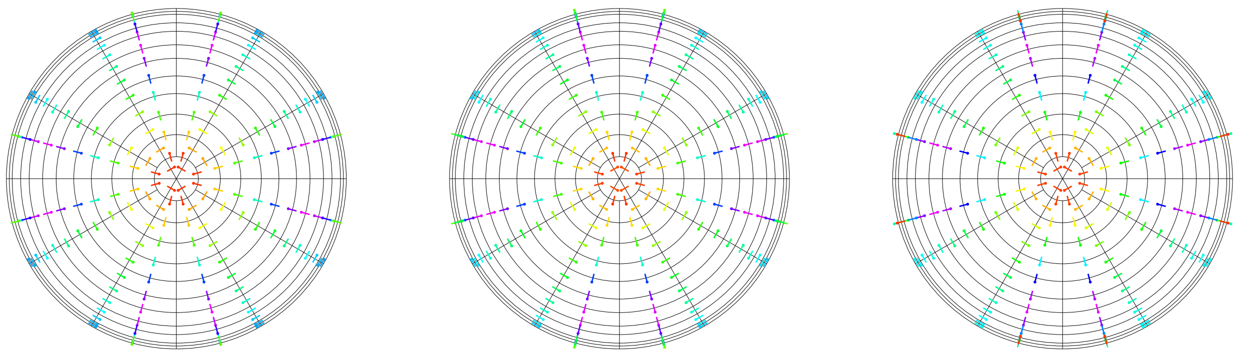
(a) Exact (b) Primal weak (c) Mixed weak

Figure 29: `diffusion/steady_state/continuous_2d_d04_p00` (Example 23.12): solutions for flow rate on mesh `hemisphere_polar_4_3_forman`



(a) Exact (b) Primal weak (c) Mixed weak

Figure 30: `diffusion/steady_state/continuous_2d_d04_p00` (Example 23.12): solutions for potential on mesh `hemisphere_polar_6_6_forman`



(a) Exact (b) Primal weak (c) Mixed weak

Figure 31: `diffusion/steady_state/continuous_2d_d04_p00` (Example 23.12): solutions for flow rate on mesh `hemisphere_polar_6_6_forman`

Example 23.13. Consider the steady-state continuous heat transport problem (Formulation 21.4, Formulation 21.7, Formulation 21.10) with input data `2d_d04_p01` in the nomenclature of the C codebase.
Concretely, $X = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ z \geq 0\right\}$ be a hemisphere with the induced metric, $\tilde{\kappa} \equiv 2$, $f = 6\kappa(x^2 - y^2)\,\mathrm{vol}$, $G_D = \{(x, y, z) \in \partial X \mid y \leq 0\}$, $G_N = \{(x, y, z) \in \partial X \mid y \geq 0\}$, $g_D(x, y, z) = x^2 - y^2$, $g_N \equiv 0$.

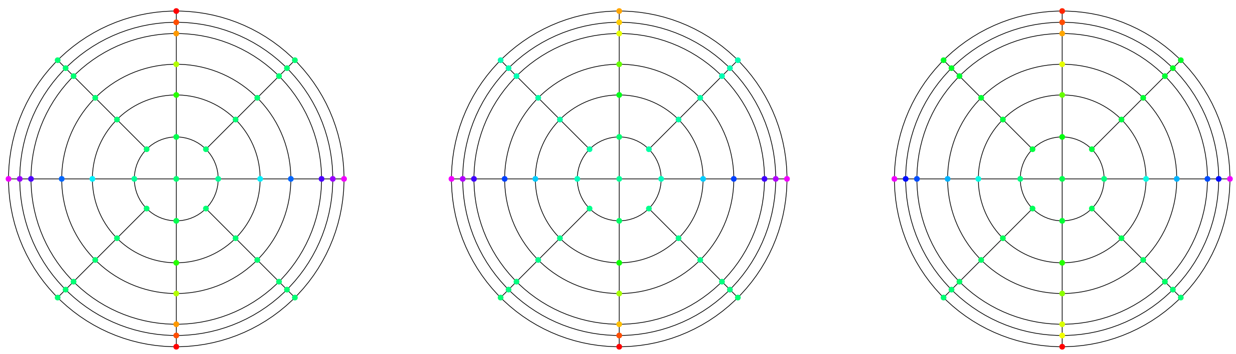
Use spherical coordinates

$$(x, y, z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),\ 0 \le \theta \le \pi/2,\ 0 \le \varphi \le 2\pi. \quad (23.14)$$

This problem has the following exact solution:

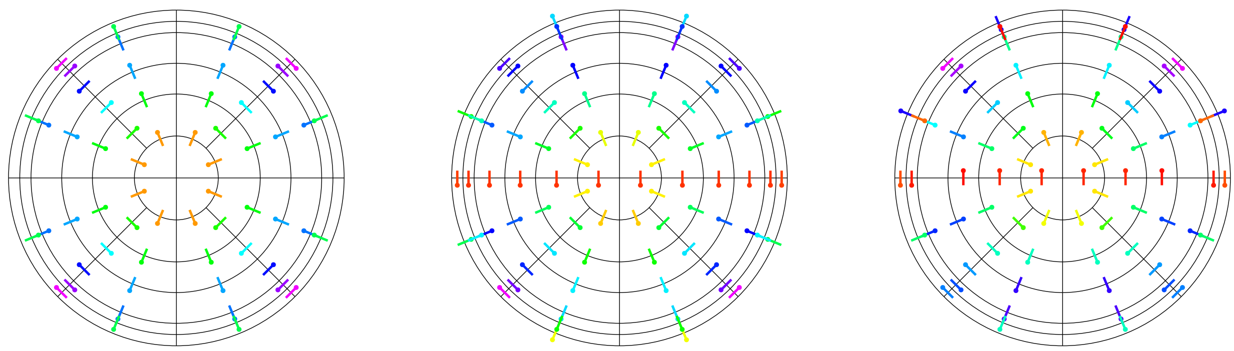
$$u(x, y, z) = x^2 - y^2, \quad (23.15a)$$
$$\tilde{q}(\theta, \varphi) = 2\kappa(-2\sin\theta\sin(2\varphi)\, d\theta - \sin\theta\sin(2\theta)\cos(2\varphi)\, d\varphi). \quad (23.15b)$$

Consider a mesh M for X consisting of n_a meridians and n_d parallels with Forman subdivision K. Its potential and flow rate on the xy-projection of K consisting of the exact solution and the 2 of the discussed cochain methods (no primal strong) are shown on Figure 32, Figure 33 ($(n_a, n_d) = (4, 3)$) and Figure 34, Figure 35 ($(n_a, n_d) = (6, 6)$).



(a) Exact (b) Primal weak (c) Mixed weak

Figure 32: `diffusion/steady_state/continuous_2d_d04_p01` (Example 23.13): solutions for potential on mesh `hemisphere_polar_4_3_forman`



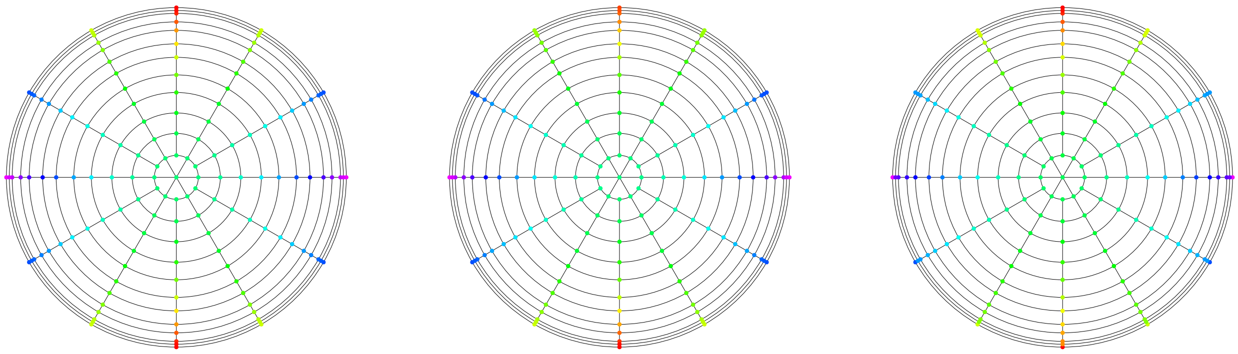
(a) Exact (b) Primal weak (c) Mixed weak

Figure 33: `diffusion/steady_state/continuous_2d_d04_p01` (Example 23.13): solutions for flow rate on mesh `hemisphere_polar_4_3_forman`

Example 23.14. Consider the steady-state continuous heat transport problem (Formulation 21.4, Formulation 21.7, Formulation 21.10) with input data `2d_d04_p02` in the nomenclature of the C codebase.
Let $X = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ z \ge 0\}$ be a hemisphere with the induced metric. Use spherical coordinates

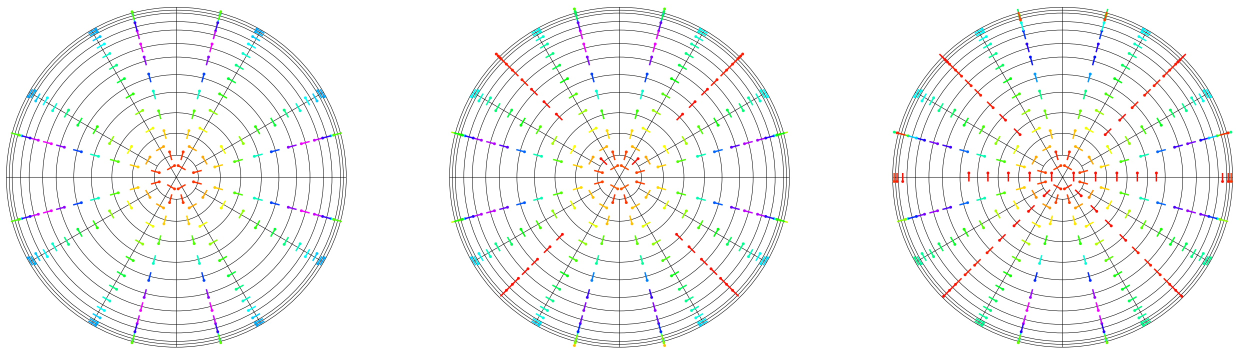
$$(x, y, z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),\ 0 \le \theta \le \pi/2,\ 0 \le \varphi \le 2\pi. \quad (23.16)$$

The input data is given in spherical coordinates as follows: $\tilde{\kappa} \equiv 2$, $\tilde{f}(\theta, \phi) = -\kappa\cos\theta\, d\theta \wedge d\phi$, $G_D = \partial X$, $G_N = \emptyset$, $\tilde{g}_D(\theta, \phi) = \pi/2$.



(a) Exact (b) Primal weak (c) Mixed weak

Figure 34: `diffusion/steady_state/continuous_2d_d04_p01` (Example 23.13): solutions for potential on mesh `hemisphere_polar_6_6_forman`



(a) Exact (b) Primal weak (c) Mixed weak

Figure 35: `diffusion/steady_state/continuous_2d_d04_p01` (Example 23.13): solutions for flow rate on mesh `hemisphere_polar_6_6_forman`

This problem has the following exact solution:

$$u(\theta, \phi) = \theta, \tag{23.17a}$$

$$\tilde{q}(\theta, \varphi) = -\kappa \sin\theta \, d\phi \tag{23.17b}$$

Consider a mesh M for X consisting of n_a meridians and n_d parallels with Forman subdivision K. Its potential and flow rate on the xy-projection of K consisting of the exact solution and the 2 of the discussed cochain methods (no primal strong) are shown on Figure 36, Figure 37 ($(n_a, n_d) = (4, 3)$) and Figure 38, Figure 39 ($(n_a, n_d) = (6, 6)$).

Example 23.15. Consider the steady-state continuous heat transport problem (Formulation 21.4, Formulation 21.7, Formulation 21.10) with input data `2d_d04_p03` in the nomenclature of the C codebase.
Let $X = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ z \geq 0\}$ be a hemisphere with the induced metric. Use spherical coordinates

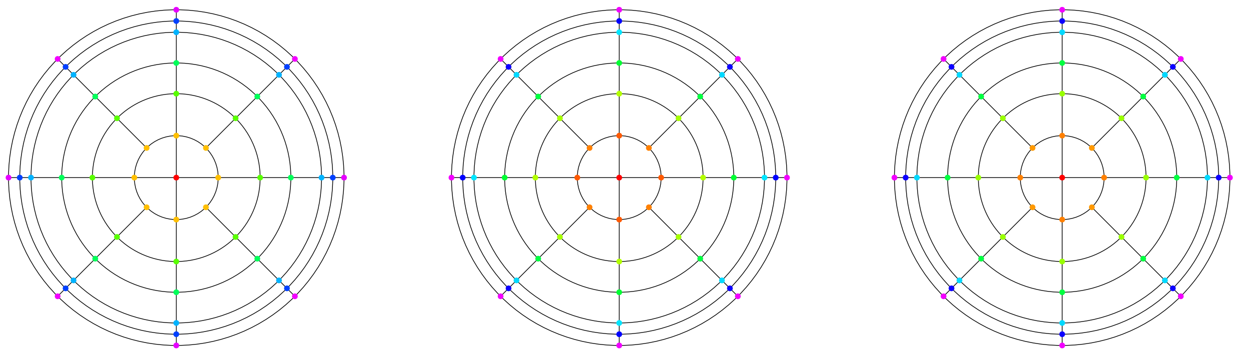
$$(x, y, z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),\ 0 \leq \theta \leq \pi/2,\ 0 \leq \varphi \leq 2\pi. \tag{23.18}$$

The input data is given in spherical coordinates as follows: $\tilde{\kappa} \equiv 2$, $\tilde{f}(\theta, \phi) = -\kappa \cos\theta \, d\theta \wedge d\phi$, $G_D = \{(x, y, z) \in \partial X \mid y \leq 0\}$, $G_N = \{(x, y, z) \in \partial X \mid y \geq 0\}$, $\widetilde{g_D}(\theta, \phi) = \pi/2$, $\widetilde{g_N}(\theta, \phi) = -\kappa \, d\phi$.
This problem has the following exact solution:

$$u(\theta, \phi) = \theta, \tag{23.19a}$$

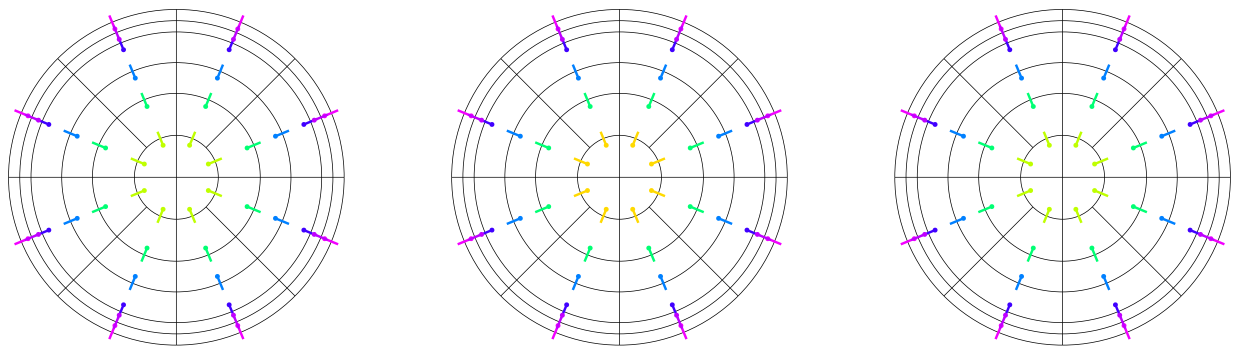
$$\tilde{q}(\theta, \varphi) = -\kappa \sin\theta \, d\phi \tag{23.19b}$$

Consider a mesh M for X consisting of n_a meridians and n_d parallels with Forman subdivision K. Its potential and flow rate on the xy-projection of K consisting of the exact solution and the 2 of the discussed cochain methods (no primal strong) are shown on Figure 40, Figure 41 ($(n_a, n_d) = (4, 3)$) and Figure 42, Figure 43 ($(n_a, n_d) = (6, 6)$).



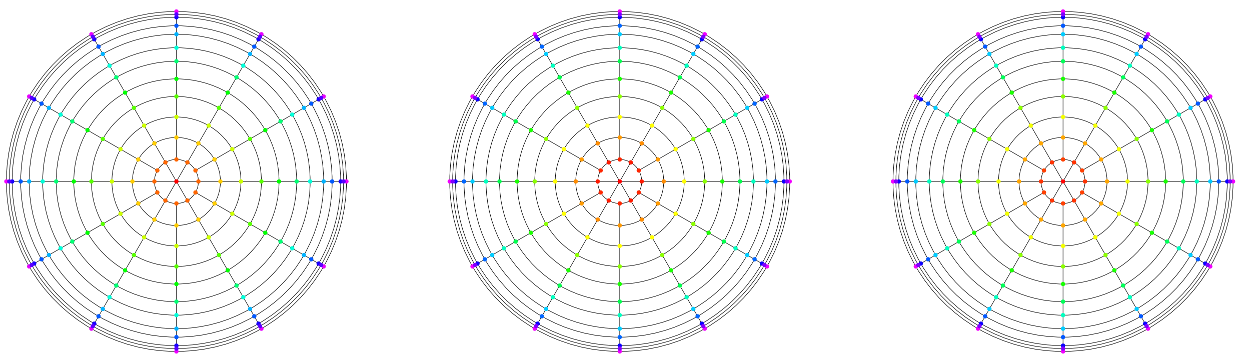
(a) Exact (b) Primal weak (c) Mixed weak

Figure 36: `diffusion/steady_state/continuous_2d_d04_p02` (Example 23.14): solutions for potential on mesh `hemisphere_polar_4_3_forman`



(a) Exact (b) Primal weak (c) Mixed weak

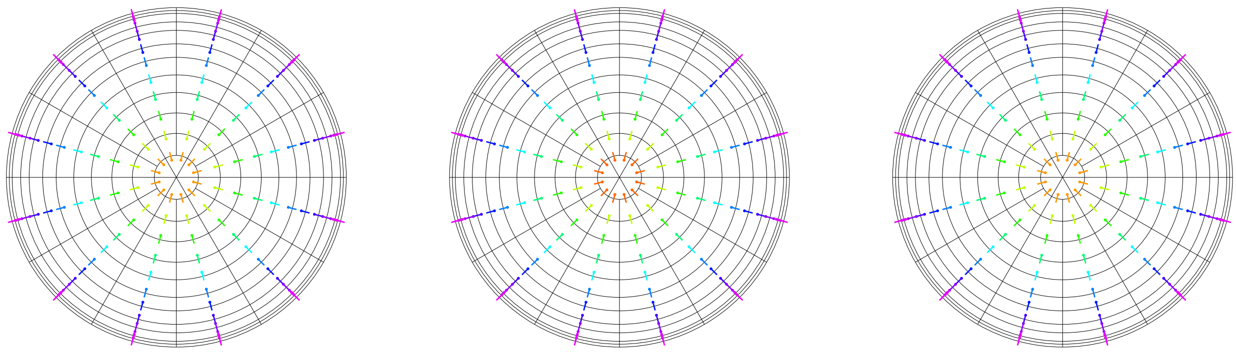
Figure 37: `diffusion/steady_state/continuous_2d_d04_p02` (Example 23.14): solutions for flow rate on mesh `hemisphere_polar_4_3_forman`



(a) Exact (b) Primal weak (c) Mixed weak

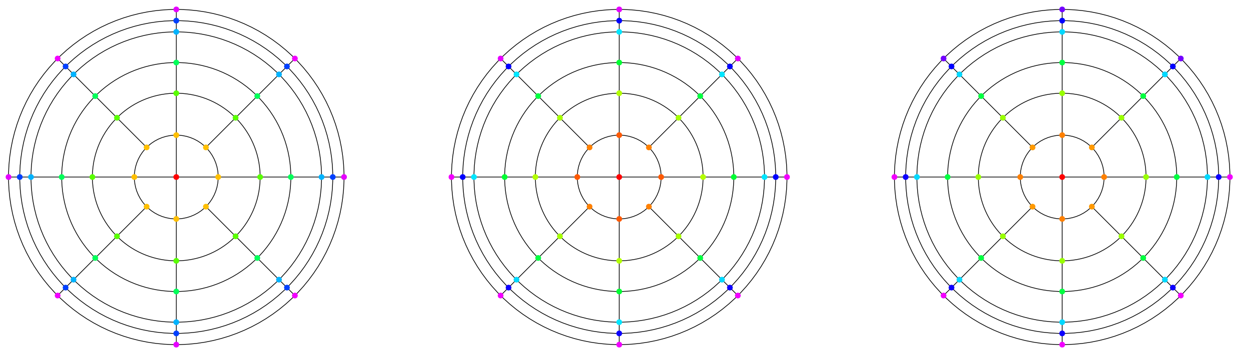
Figure 38: `diffusion/steady_state/continuous_2d_d04_p02` (Example 23.14): solutions for potential on mesh `hemisphere_polar_6_6_forman`

Example 23.16. Consider the steady-state continuous heat transport problem (Formulation 21.4, Formulation 21.7, Formulation 21.10) with input data `2d_parallelogram_20_15_degrees_45_p00` in the nomenclature of the C codebase.
Concretely, $a = \sqrt{15}$, $X = \text{Polygon}((0,0),(20,0),(20+\text{a},\text{a}),(\text{a},\text{a}))$, $\tilde{\kappa} \equiv 1$, $f \equiv 0$, $G_D = \text{Line}((0,0),(a,a)) \cup \text{Line}((20,0),(20+a,a))$, $G_N = \text{Line}((0,0),(20,0)) \cup \text{Line}((a,a),(20+a,a))$, $g_D(x,y) =$



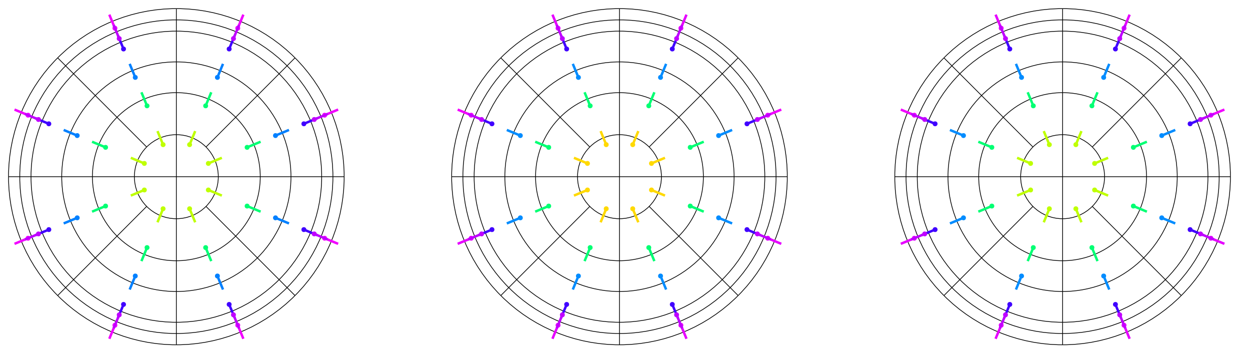
(a) Exact (b) Primal weak (c) Mixed weak

Figure 39: `diffusion/steady_state/continuous_2d_d04_p02` (Example 23.14): solutions for flow rate on mesh `hemisphere_polar_6_6_forman`



(a) Exact (b) Primal weak (c) Mixed weak

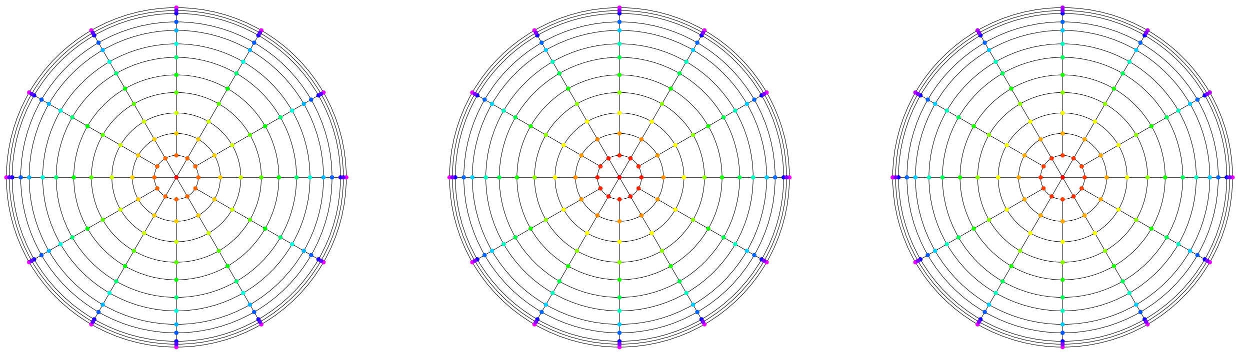
Figure 40: `diffusion/steady_state/continuous_2d_d04_p03` (Example 23.15): solutions for potential on mesh `hemisphere_polar_4_3_forman`



(a) Exact (b) Primal weak (c) Mixed weak

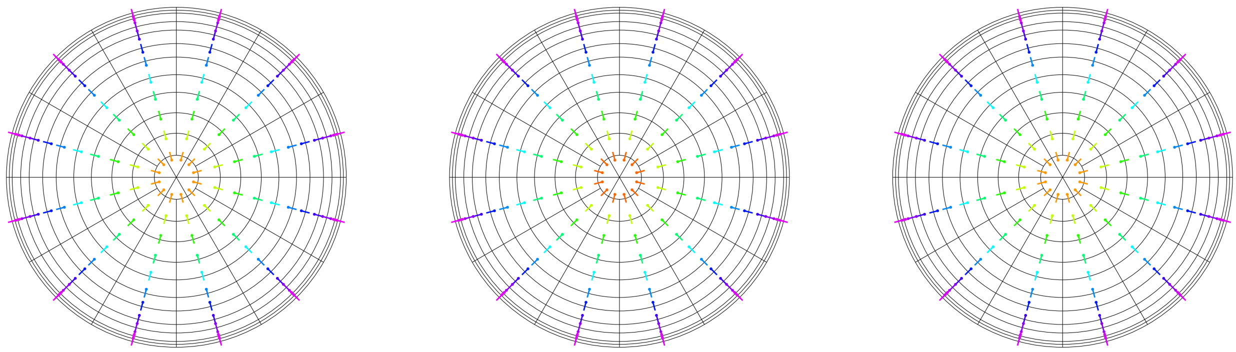
Figure 41: `diffusion/steady_state/continuous_2d_d04_p03` (Example 23.15): solutions for flow rate on mesh `hemisphere_polar_4_3_forman`

$$\begin{cases} 20, & (x, y) \in \mathrm{Line}((0,0),(a,a)) \\ 0, & (x, y) \in \mathrm{Line}((20,0),(20+a,a)) \end{cases}, \; g_N = dx.$$



(a) Exact (b) Primal weak (c) Mixed weak

Figure 42: `diffusion/steady_state/continuous_2d_d04_p03` (Example 23.15): solutions for potential on mesh `hemisphere_polar_6_6_forman`



(a) Exact (b) Primal weak (c) Mixed weak

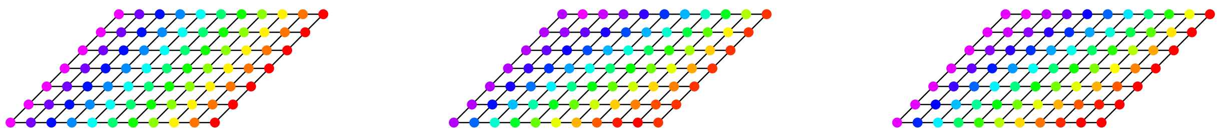
Figure 43: `diffusion/steady_state/continuous_2d_d04_p03` (Example 23.15): solutions for flow rate on mesh `hemisphere_polar_6_6_forman`

This problem has the following exact solution:

$$u(x, y) = 20 - (x - y), \tag{23.20a}$$

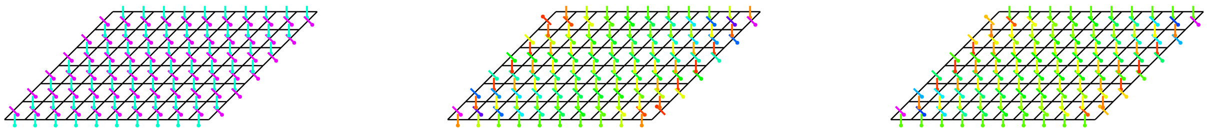
$$q = dx + dy. \tag{23.20b}$$

Consider a regular mesh M for X consisting of 5×3 parallelograms with Forman subdivision K (10×6 parallelograms). Its potential and flow rate on K consisting of the exact solution and the mixed weak method are shown on Figure 44 and Figure 45.



(a) Exact (b) Primal weak (c) Mixed weak

Figure 44: `diffusion/steady_state/continuous_2d_parallelogram_20_15_degrees_45_p00` (Example 23.16): solutions for potential



(a) Exact (b) Primal weak (c) Mixed weak

Figure 45: `diffusion/steady_state/continuous_2d_parallelogram_20_15_degrees_45_p00` (Example 23.16): solutions for flow rate

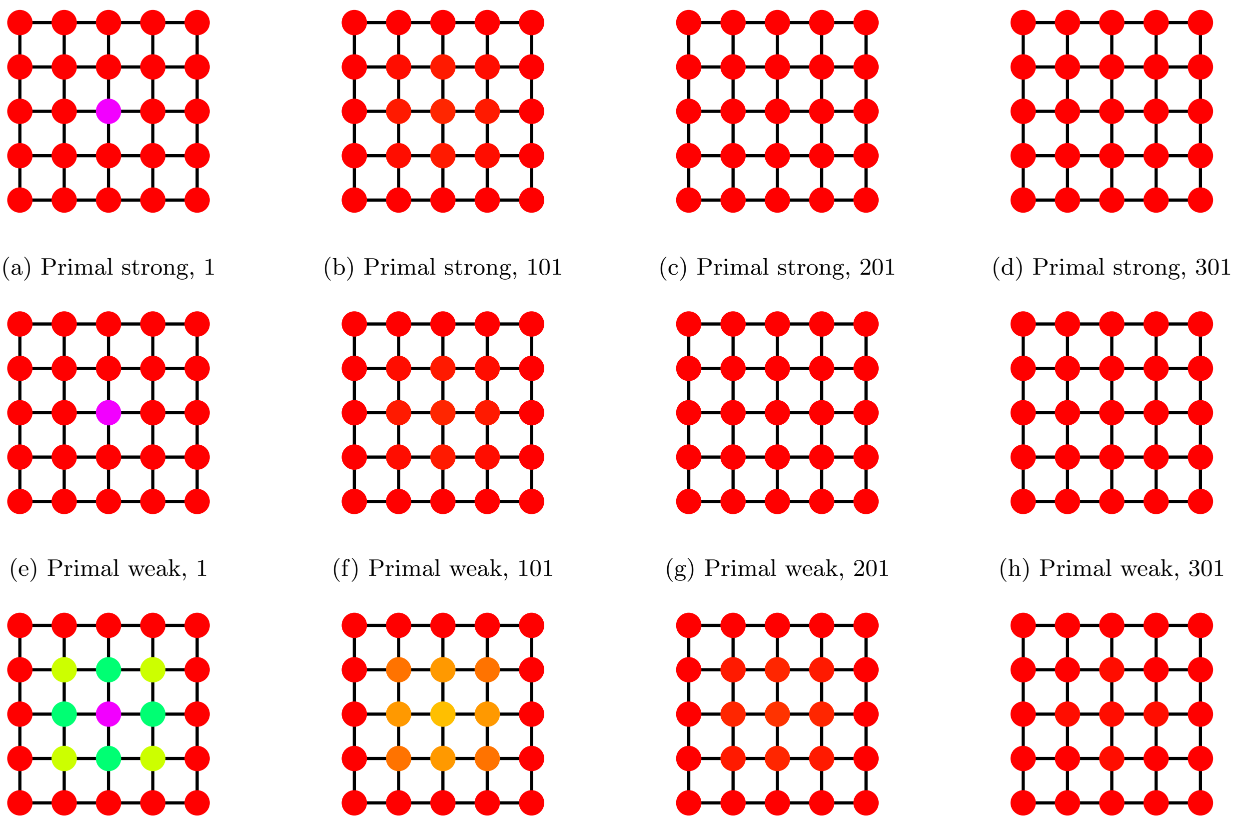
23.2 Transient

Example 23.17. Consider the transient continuous heat transport problem (Formulation 21.3, Formulation 21.6, Formulation 21.9) with input data `2d_d00_p00` in the nomenclature of the C codebase.

Concretely, $X = [0,1]^2$, $\tilde{\pi} \equiv 1$, $\tilde{\kappa} \equiv 1$, $u_0(x,y) = \begin{cases} 100, & (x,y) = (0.5, 0.5) \\ 0, & \text{else} \end{cases}$, $f \equiv 0$, $G_D = \partial X$, $G_N = \emptyset$, $g_D(x,y) = 0$.
This problem has the following exact solution in steady-state:

$$u \equiv 0, \tag{23.21a}$$
$$q \equiv 0. \tag{23.21b}$$

Consider a mesh M for X consisting of 2×2 squares (each axis is divided into 2 segments) with Forman subdivision K (4×4 squares). Its potential and flow rate on K consisting of the 3 discussed cochain methods are shown on Figure 46 and Figure 47.

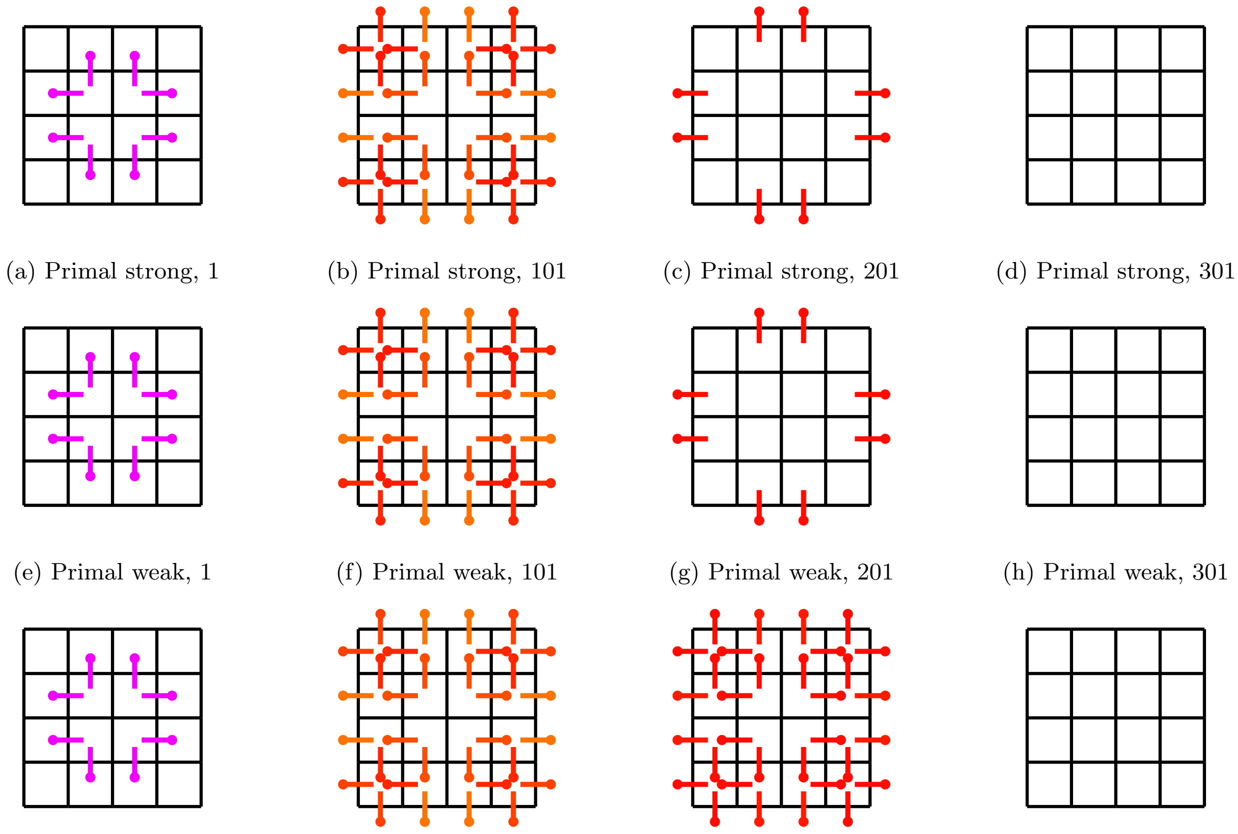


(a) Primal strong, 1 (b) Primal strong, 101 (c) Primal strong, 201 (d) Primal strong, 301

(e) Primal weak, 1 (f) Primal weak, 101 (g) Primal weak, 201 (h) Primal weak, 301

(i) Mixed weak, 1 (j) Mixed weak, 101 (k) Mixed weak, 201 (l) Mixed weak, 301

Figure 46: `diffusion/transient/continuous_2d_d00_p00` (Example 23.17): solutions for potential



(a) Primal strong, 1 (b) Primal strong, 101 (c) Primal strong, 201 (d) Primal strong, 301

(e) Primal weak, 1 (f) Primal weak, 101 (g) Primal weak, 201 (h) Primal weak, 301

(i) Mixed weak, 1 (j) Mixed weak, 101 (k) Mixed weak, 201 (l) Mixed weak, 301

Figure 47: `diffusion/transient/continuous_2d_d00_p00` (Example 23.17): solutions for flow rate

Example 23.18. Consider the transient continuous heat transport problem (Formulation 21.3, Formulation 21.6, Formulation 21.9) with input data `2d_d00_p01` in the nomenclature of the C codebase.
Concretely, $X = [0,1]^2$, $\tilde{\pi} \equiv 4$, $\tilde{\kappa} \equiv 1$, $u_0(x, y) = \begin{cases} 100, & x = 1 \\ -100, & \text{else} \end{cases}$, $f \equiv 0$, $G_D = \{0, 1\} \times [0, 1]$, $G_N = [0, 1] \times \{0, 1\}$, $g_D(x, y) = \begin{cases} 100, & x = 1 \\ -100, & x = 0 \end{cases}$, $g_N \equiv 0$.
This problem has the following exact solution in steady-state:

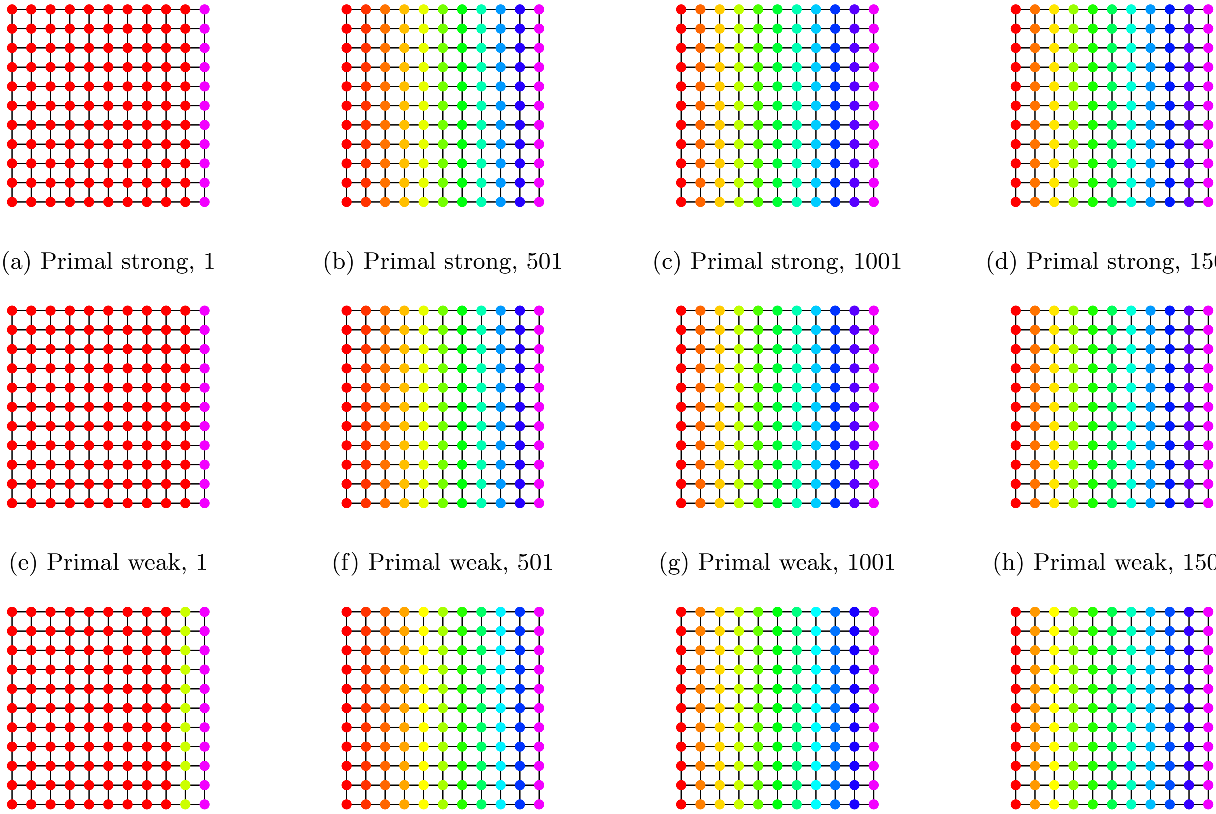
$$u(x, y) = 100(2x - 1), \tag{23.22a}$$
$$q = -200\, dy. \tag{23.22b}$$

Consider a mesh M for X consisting of 5×5 squares (each axis is divided into 5 segments) with Forman subdivision K (10×10 squares). Its potential and flow rate on K consisting of the 3 discussed cochain methods are shown on Figure 48 and Figure 49.

Example 23.19. Consider the transient continuous heat transport problem (Formulation 21.3, Formulation 21.6, Formulation 21.9) with input data `2d_d00_p02` in the nomenclature of the C codebase.
Concretely, $X = [0,1]^2$, $\tilde{\pi} \equiv 1$, $\tilde{\kappa} \equiv 1$, $u_0(x, y) = x^2 + y^2$, $f \equiv -4\, dx \wedge dy$, $G_D = \partial X$, $G_N = \emptyset$, $g_D(x, y) = x^2 + y^2$.
This problem is in steady-state mode from the beginning with the following exact solution:

$$u(x, y) = x^2 + y^2, \tag{23.23a}$$
$$q(x, y) = 2y\, dx - 2x\, dy. \tag{23.23b}$$

Consider a mesh M for X consisting of 2×2 squares (each axis is divided into 2 segments) with Forman subdivision K (4×4 squares). Its potential and flow rate on K consisting of the 3 discussed cochain methods are shown on Figure 50 and Figure 51.

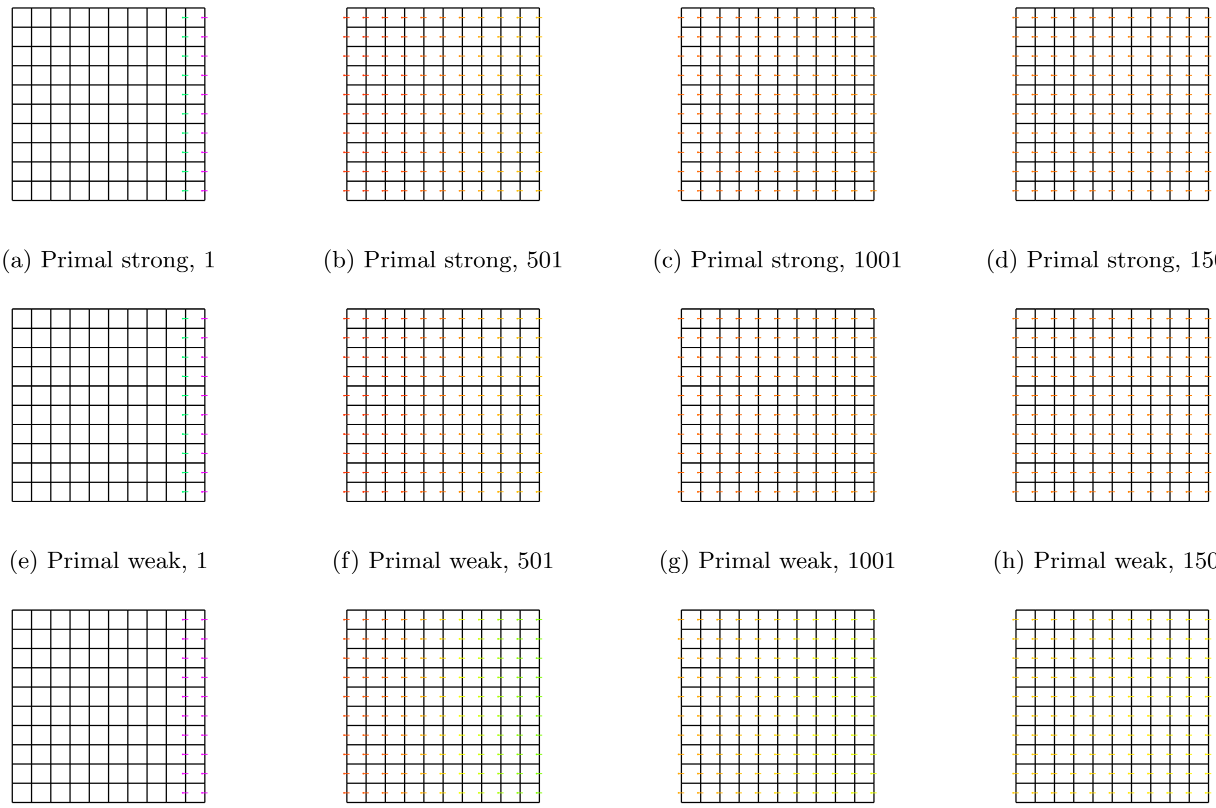


(a) Primal strong, 1 (b) Primal strong, 501 (c) Primal strong, 1001 (d) Primal strong, 1501

(e) Primal weak, 1 (f) Primal weak, 501 (g) Primal weak, 1001 (h) Primal weak, 1501

(i) Mixed weak, 1 (j) Mixed weak, 501 (k) Mixed weak, 1001 (l) Mixed weak, 1501

Figure 48: `diffusion/transient/continuous_2d_d00_p01` (Example 23.18): solutions for potential

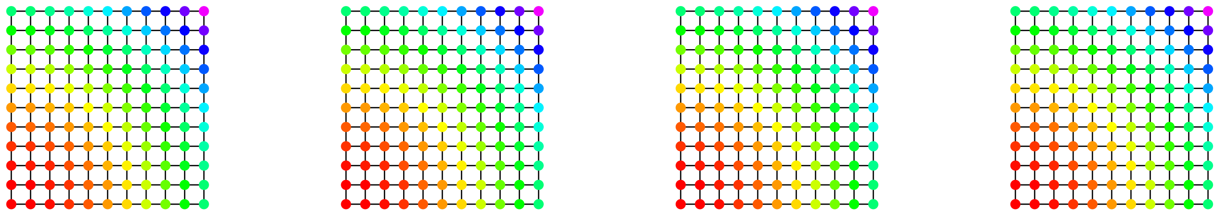


(a) Primal strong, 1 (b) Primal strong, 501 (c) Primal strong, 1001 (d) Primal strong, 1501

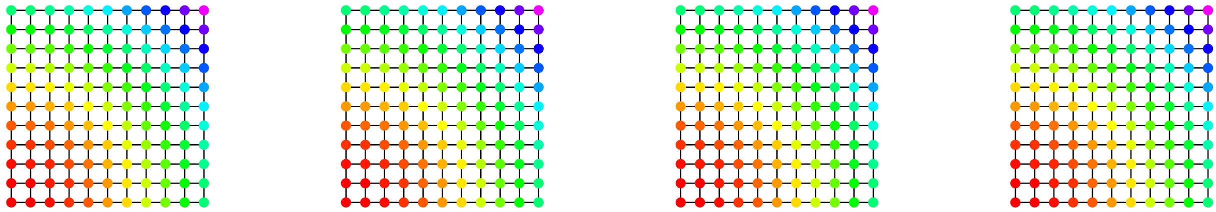
(e) Primal weak, 1 (f) Primal weak, 501 (g) Primal weak, 1001 (h) Primal weak, 1501

(i) Mixed weak, 1 (j) Mixed weak, 501 (k) Mixed weak, 1001 (l) Mixed weak, 1501

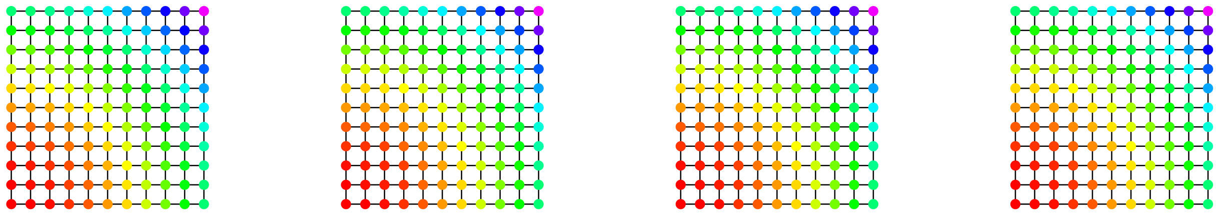
Figure 49: `diffusion/transient/continuous_2d_d00_p01` (Example 23.18): solutions for flow rate



(a) Primal strong, 1 (b) Primal strong, 334 (c) Primal strong, 667 (d) Primal strong, 1001

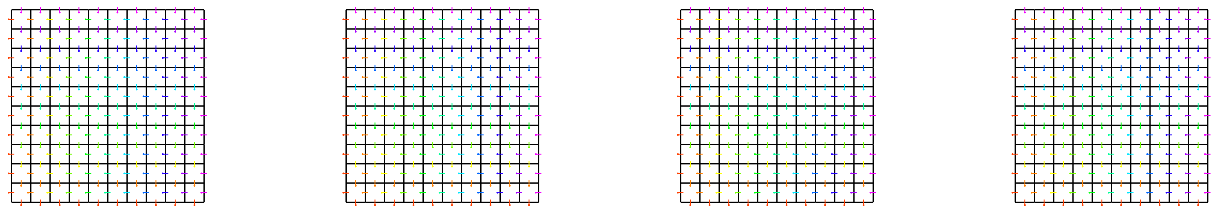


(e) Primal weak, 1 (f) Primal weak, 334 (g) Primal weak, 667 (h) Primal weak, 1001

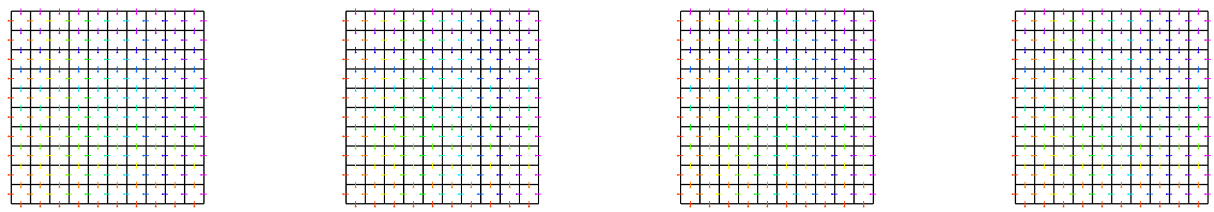


(i) Mixed weak, 1 (j) Mixed weak, 334 (k) Mixed weak, 667 (l) Mixed weak, 1001

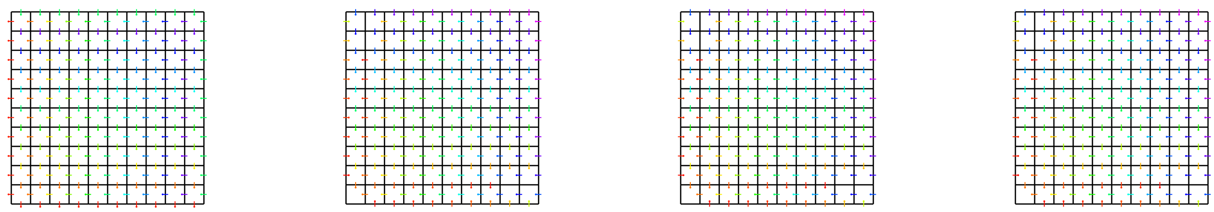
Figure 50: `diffusion/transient/continuous_2d_d00_p02` (Example 23.19): solutions for potential



(a) Primal strong, 1 (b) Primal strong, 334 (c) Primal strong, 667 (d) Primal strong, 1001



(e) Primal weak, 1 (f) Primal weak, 334 (g) Primal weak, 667 (h) Primal weak, 1001



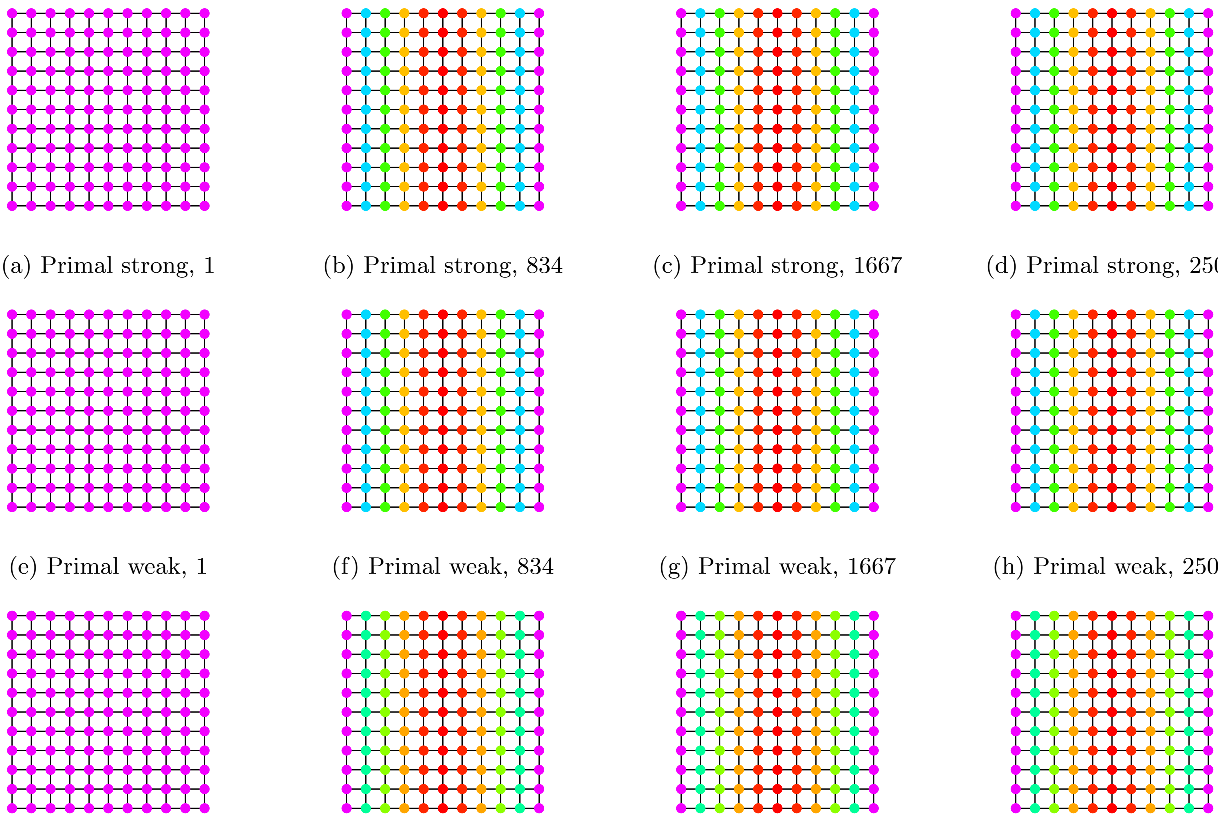
(i) Mixed weak, 1 (j) Mixed weak, 334 (k) Mixed weak, 667 (l) Mixed weak, 1001

Figure 51: `diffusion/transient/continuous_2d_d00_p02` (Example 23.19): solutions for flow rate

Example 23.20. Consider the transient continuous heat transport problem (Formulation 21.3, Formulation 21.6, Formulation 21.9) with input data `2d_d00_p03` in the nomenclature of the C codebase.
Concretely, $X = [0,1]^2$, $\tilde{\pi} \equiv 4$, $\tilde{\kappa} \equiv 1$, $u_0(x,y) = 0$, $f \equiv -2\,dx \wedge dy$, $G_D = \{0,1\} \times [0,1]$, $G_N = [0,1] \times \{0,1\}$, $g_D(x,y) = 0$, $g_N \equiv 0$.
This problem has the following exact solution in steady-state:

$$u(x,y) = x(x-1), \tag{23.24a}$$
$$q(x,y) = -(2x-1)\,dy. \tag{23.24b}$$

Consider a mesh M for X consisting of 5×5 squares (each axis is divided into 5 segments) with Forman subdivision K (10×10 squares). Its potential and flow rate on K consisting of the 3 discussed cochain methods are shown on Figure 52 and Figure 53.



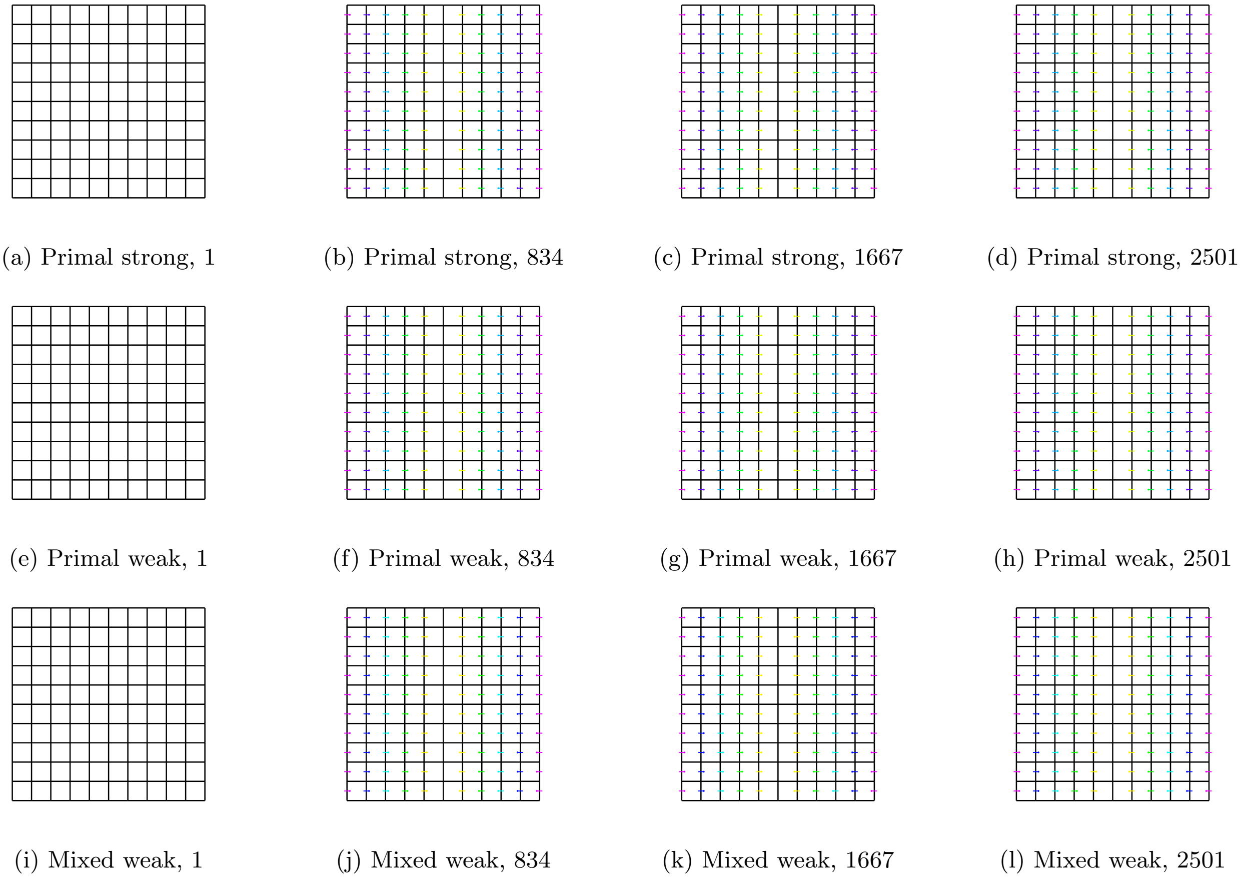
(a) Primal strong, 1; (b) Primal strong, 834; (c) Primal strong, 1667; (d) Primal strong, 2501
(e) Primal weak, 1; (f) Primal weak, 834; (g) Primal weak, 1667; (h) Primal weak, 2501
(i) Mixed weak, 1; (j) Mixed weak, 834; (k) Mixed weak, 1667; (l) Mixed weak, 2501

Figure 52: `diffusion/transient/continuous_2d_d00_p03` (Example 23.20): solutions for potential

Example 23.21. Consider the transient continuous heat transport problem (Formulation 21.3, Formulation 21.6, Formulation 21.9) with input data `2d_d00_p04` in the nomenclature of the C codebase.
Concretely, $X = [0,1]^2$, $\tilde{\pi} \equiv 4$, $\tilde{\kappa} \equiv 1$, $u_0(x,y) = y(y-1)$, $f \equiv -4\,dx \wedge dy$, $G_D = \{0,1\} \times [0,1]$, $G_N = [0,1] \times \{0,1\}$, $g_D(x,y) = y(y-1)$, $g_N(x,y) = (2y-1)\,dx = \begin{cases} -dx, & y = 0 \\ dx, & y = 1 \end{cases}$.
This problem has the following exact solution in steady-state:

$$u(x,y) = x(x-1) + y(y-1), \tag{23.25a}$$
$$q(x,y) = (2y-1)\,dx - (2x-1)\,dy. \tag{23.25b}$$

Consider a mesh M for X consisting of 5×5 squares (each axis is divided into 5 segments) with Forman subdivision K (10×10 squares). Its potential and flow rate on K consisting of the 3 discussed cochain methods are shown on Figure 54 and Figure 55.

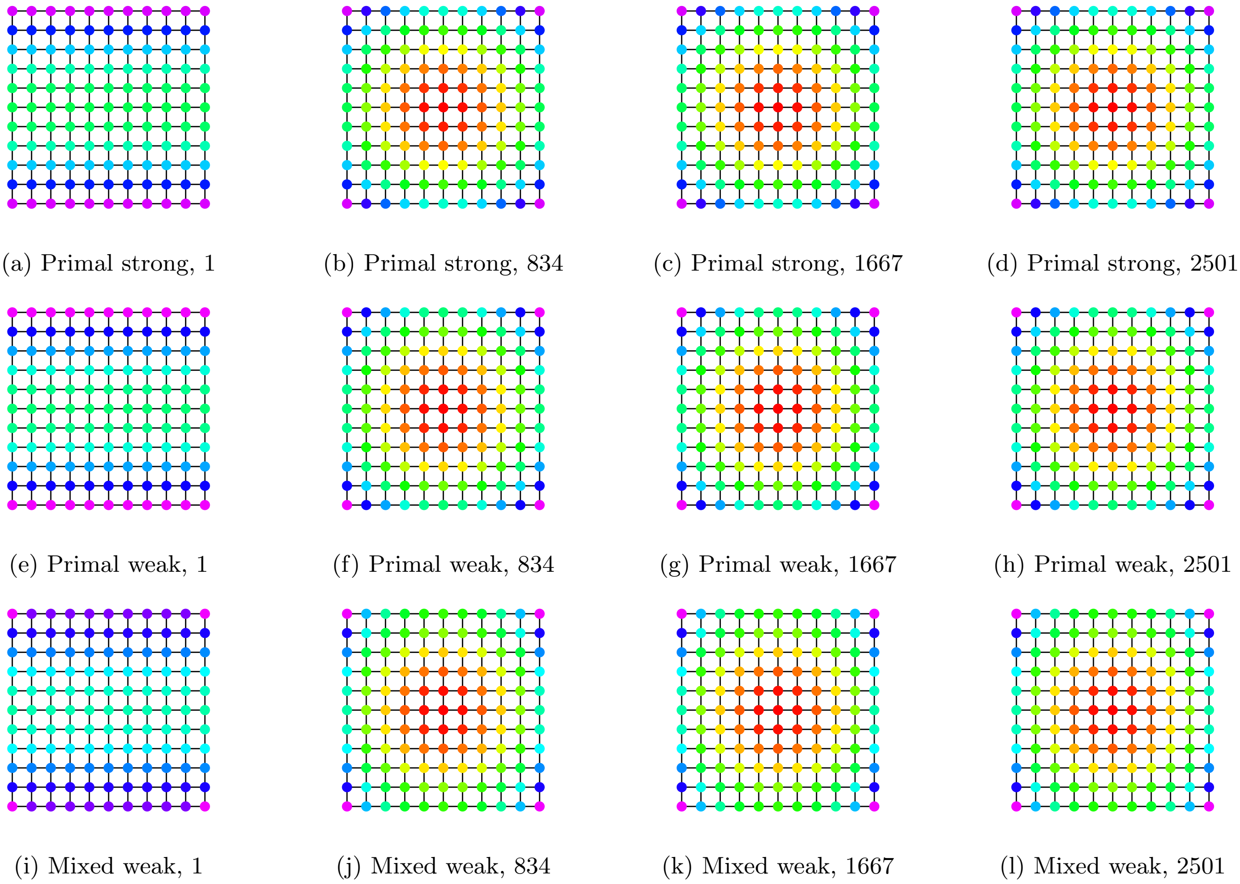


(a) Primal strong, 1 (b) Primal strong, 834 (c) Primal strong, 1667 (d) Primal strong, 2501

(e) Primal weak, 1 (f) Primal weak, 834 (g) Primal weak, 1667 (h) Primal weak, 2501

(i) Mixed weak, 1 (j) Mixed weak, 834 (k) Mixed weak, 1667 (l) Mixed weak, 2501

Figure 53: `diffusion/transient/continuous_2d_d00_p03` (Example 23.20): solutions for flow rate

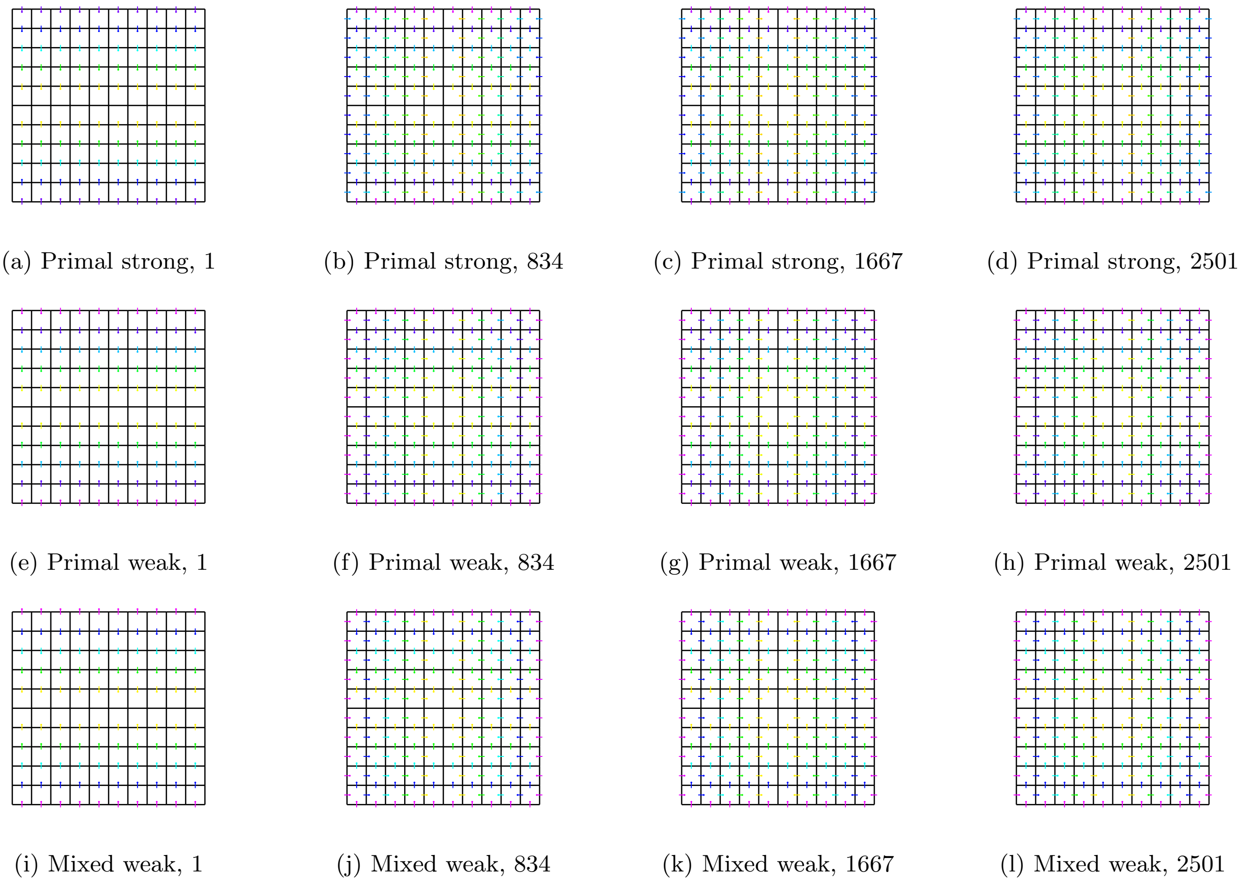


(a) Primal strong, 1 (b) Primal strong, 834 (c) Primal strong, 1667 (d) Primal strong, 2501

(e) Primal weak, 1 (f) Primal weak, 834 (g) Primal weak, 1667 (h) Primal weak, 2501

(i) Mixed weak, 1 (j) Mixed weak, 834 (k) Mixed weak, 1667 (l) Mixed weak, 2501

Figure 54: `diffusion/transient/continuous_2d_d00_p04` (Example 23.21): solutions for potential



(a) Primal strong, 1 (b) Primal strong, 834 (c) Primal strong, 1667 (d) Primal strong, 2501

(e) Primal weak, 1 (f) Primal weak, 834 (g) Primal weak, 1667 (h) Primal weak, 2501

(i) Mixed weak, 1 (j) Mixed weak, 834 (k) Mixed weak, 1667 (l) Mixed weak, 2501

Figure 55: `diffusion/transient/continuous_2d_d00_p04` (Example 23.21): solutions for flow rate

Example 23.22. Consider the transient continuous heat transport problem (Formulation 21.3, Formulation 21.6, Formulation 21.9) with input data `2d_d00_p05` in the nomenclature of the C codebase.
Concretely, $X = [0,1]^2$, $\tilde{\pi} \equiv 0$, $\tilde{\kappa} \equiv 1$, $u_0(x,y) = \sin(\pi x)\sin(\pi y)$, $f \equiv 0$, $G_D = \partial X$, $G_N = \emptyset$, $g_D(x,y) = 0$.
This problem has the following exact solution:

$$u(t,x,y) = e^{-2\pi t^2} \sin(\pi x)\sin(\pi y) \tag{23.26a}$$

$$q(t,x,y) = \pi e^{-2\pi t^2} (\sin(\pi x)\cos(\pi y)\, dx - \cos(\pi x)\sin(\pi y)\, dy). \tag{23.26b}$$

Consider a mesh M for X consisting of 5×5 squares (each axis is divided into 5 segments) with Forman subdivision K (10×10 squares). Its potential and flow rate on K consisting of the 3 discussed cochain methods are shown on Figure 56 and Figure 57.

Example 23.23. Consider the transient continuous heat transport problem (Formulation 21.3, Formulation 21.6, Formulation 21.9) with input data `2d_d01_p00` in the nomenclature of the C codebase.
Concretely, $X = \text{polygon}((-5,0),(0,-5),(5,0),(0,5)$, $\tilde{\pi} \equiv 6$, $\tilde{\kappa} \equiv 6$,
$u_0(x,y) = \begin{cases} 100, & (x,y) \in \text{line}((-5,0),(0,-5)) \\ 0, & (x,y) \in \text{line}((5,0),(0,5)) \end{cases}$, $f \equiv 0$,
$G_D = \text{line}((-5,0),(0,-5)) \cup \text{line}((5,0),(0,5))$, $G_N = \text{line}((0,-5),(5,0)) \cup \text{line}((0,5),(-5,0))$,
$g_D(x,y) = \begin{cases} 100, & (x,y) \in \text{line}((-5,0),(0,-5)) \\ 0, & (x,y) \in \text{line}((5,0),(0,5)) \end{cases}$, $g_N \equiv 0$.
This problem has the following exact solution in steady-state:

$$u(x,y) = 50(1 - (x+y)/5), \tag{23.27a}$$

$$q(x,y) = 60(-dx + dy). \tag{23.27b}$$

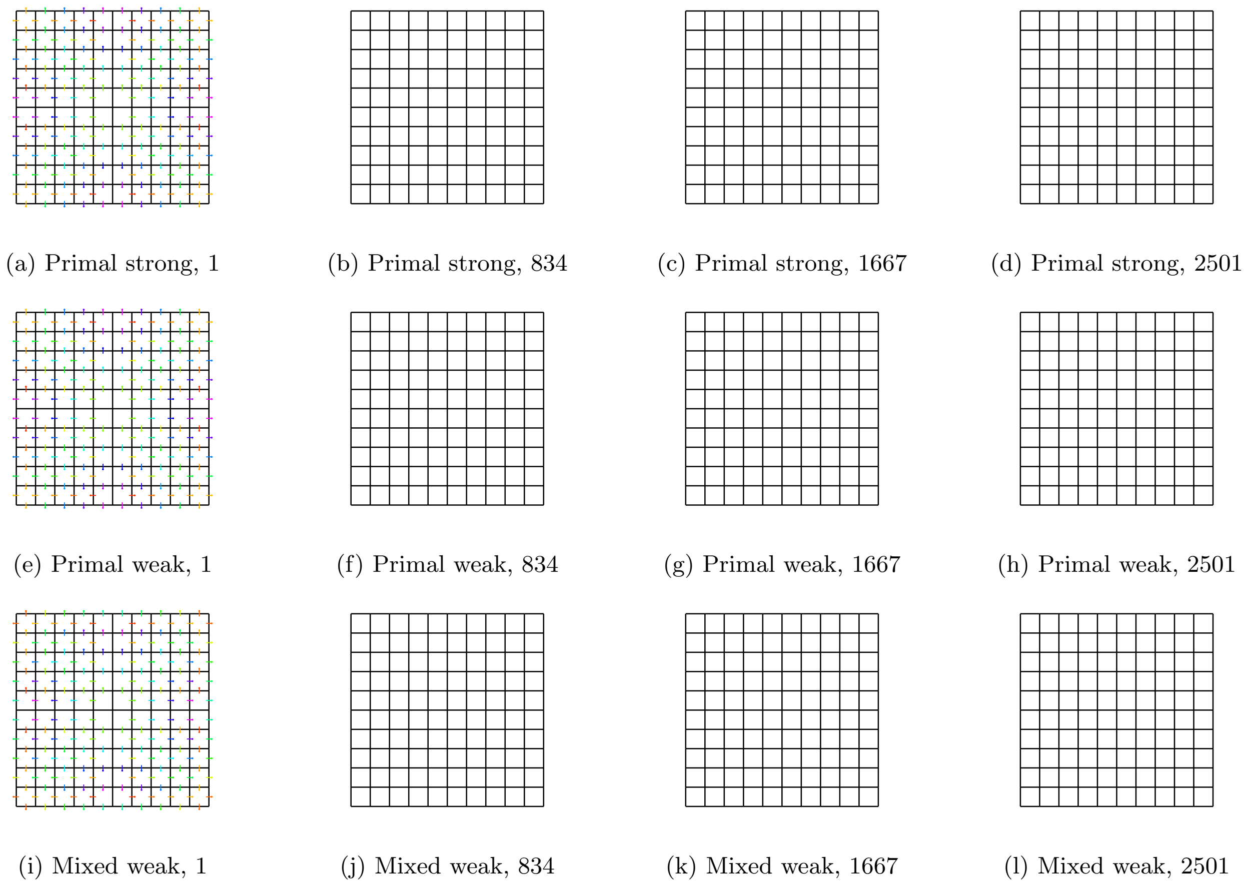
Consider a mesh M for X consisting of 4×4 squares (each axis is divided into 4 segments) with Forman subdivision K (8×8 squares). Its potential and flow rate on K consisting of the 3 discussed cochain methods are shown on Figure 58 and Figure 59.

(a) Primal strong, 1 (b) Primal strong, 834 (c) Primal strong, 1667 (d) Primal strong, 2501

(e) Primal weak, 1 (f) Primal weak, 834 (g) Primal weak, 1667 (h) Primal weak, 2501

(i) Mixed weak, 1 (j) Mixed weak, 834 (k) Mixed weak, 1667 (l) Mixed weak, 2501

Figure 56: `diffusion/transient/continuous_2d_d00_p05` (Example 23.22): solutions for potential

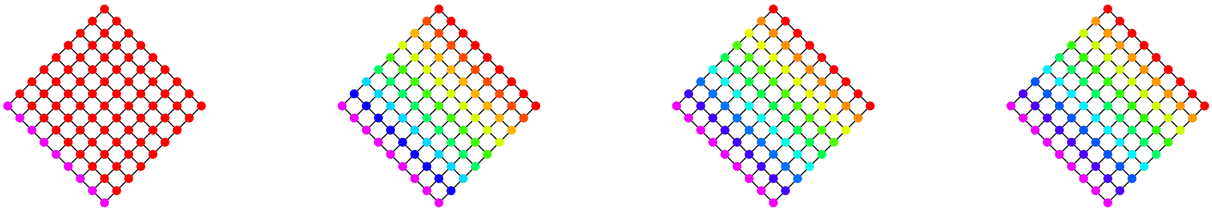


(a) Primal strong, 1 (b) Primal strong, 834 (c) Primal strong, 1667 (d) Primal strong, 2501

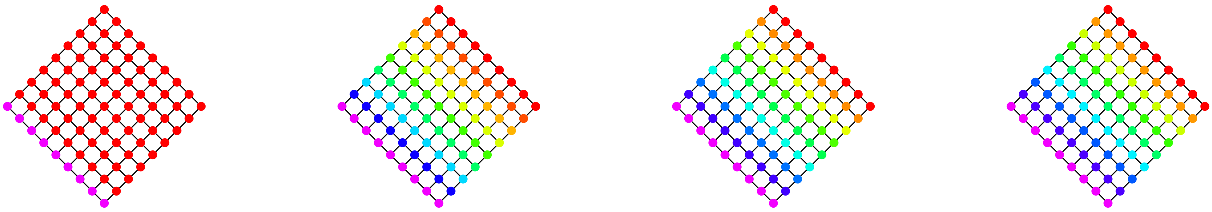
(e) Primal weak, 1 (f) Primal weak, 834 (g) Primal weak, 1667 (h) Primal weak, 2501

(i) Mixed weak, 1 (j) Mixed weak, 834 (k) Mixed weak, 1667 (l) Mixed weak, 2501

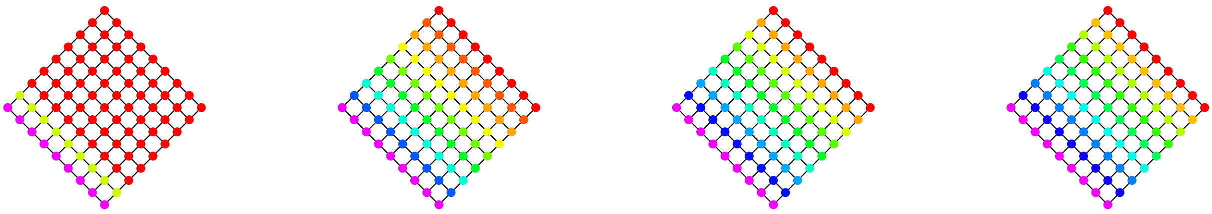
Figure 57: `diffusion/transient/continuous_2d_d00_p05` (Example 23.22): solutions for flow rate



(a) Primal strong, 1 (b) Primal strong, 101 (c) Primal strong, 201 (d) Primal strong, 301

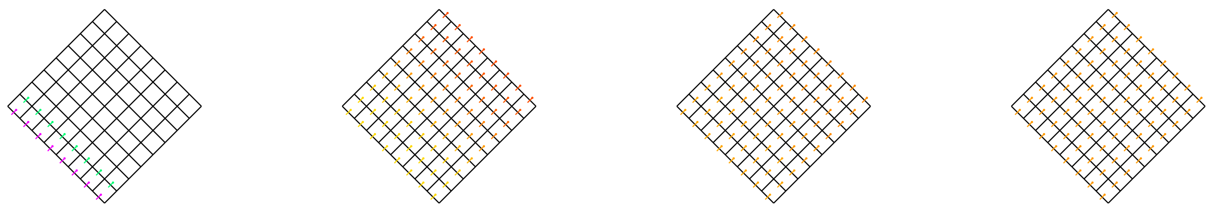


(e) Primal weak, 1 (f) Primal weak, 101 (g) Primal weak, 201 (h) Primal weak, 301

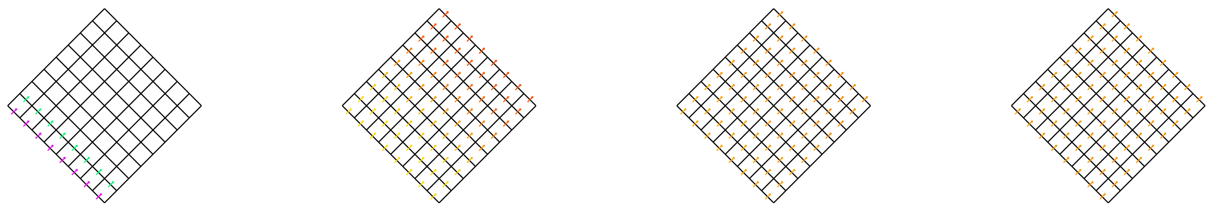


(i) Mixed weak, 1 (j) Mixed weak, 101 (k) Mixed weak, 201 (l) Mixed weak, 301

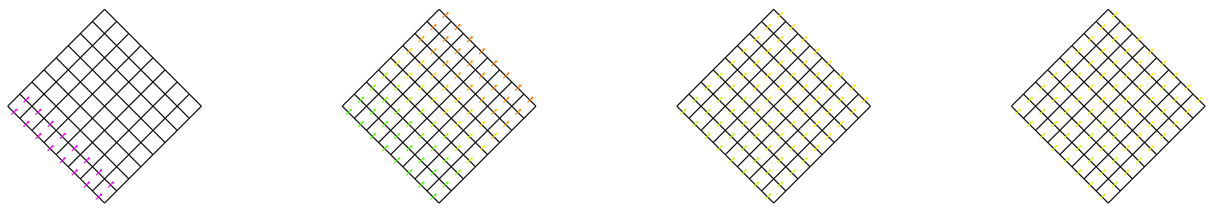
Figure 58: `diffusion/transient/continuous_2d_d01_p00` (Example 23.23): solutions for potential



(a) Primal strong, 1 (b) Primal strong, 101 (c) Primal strong, 201 (d) Primal strong, 301



(e) Primal weak, 1 (f) Primal weak, 101 (g) Primal weak, 201 (h) Primal weak, 301



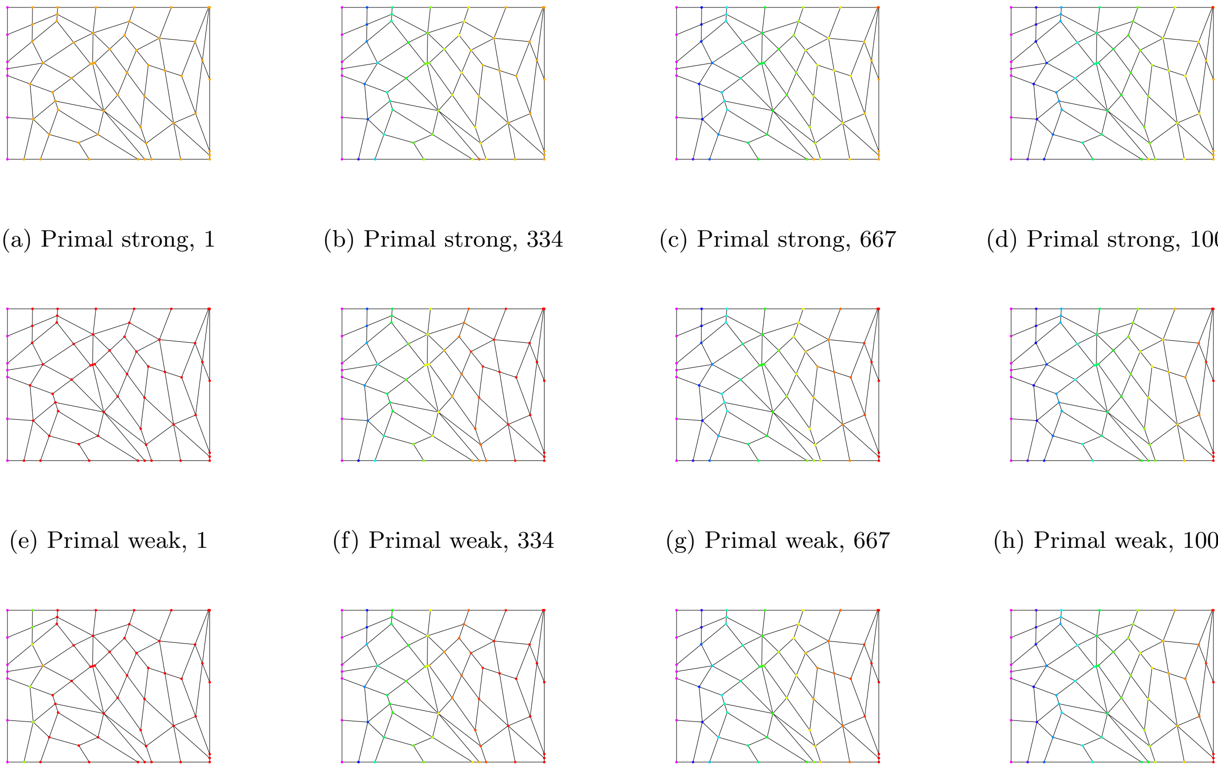
(i) Mixed weak, 1 (j) Mixed weak, 101 (k) Mixed weak, 201 (l) Mixed weak, 301

Figure 59: `diffusion/transient/continuous_2d_d01_p00` (Example 23.23): solutions for flow rate

Example 23.24. Consider the transient continuous heat transport problem (Formulation 21.3, Formulation 21.6, Formulation 21.9) with input data `2d_d02_p00` in the nomenclature of the C codebase.
Concretely, $X = [0, 20] \times [0, 15]$, $\tilde{\pi} \equiv 4$, $\tilde{\kappa} \equiv 6$, $u_0(x, y) = \begin{cases} 100, & x = 0 \\ 0, & x > 0 \end{cases}$, $f \equiv 0$, $G_D = \{0, 20\} \times [0, 15]$, $G_N = [0, 20] \times \{0, 15\}$, $g_D(x, y) = \begin{cases} 100, & x = 0 \\ 0, & x = 20 \end{cases}$, $g_N \equiv 0$.
This problem has the following exact solution in steady-state:

$$u(x, y) = 5(20 - x), \tag{23.28a}$$
$$q(x, y) = 30\, dy. \tag{23.28b}$$

For this problem I use a mesh M generated by Neper with Forman subdivision K. Its potential and flow rate on K consisting of the 3 discussed cochain methods are shown on Figure 60 and Figure 61.



(a) Primal strong, 1 (b) Primal strong, 334 (c) Primal strong, 667 (d) Primal strong, 1001

(e) Primal weak, 1 (f) Primal weak, 334 (g) Primal weak, 667 (h) Primal weak, 1001

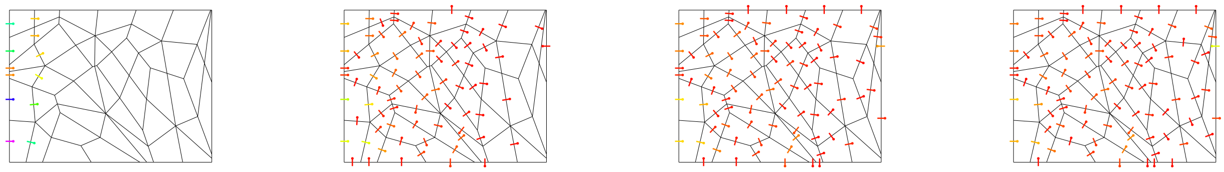
(i) Mixed weak, 1 (j) Mixed weak, 334 (k) Mixed weak, 667 (l) Mixed weak, 1001

Figure 60: `diffusion/transient/continuous_2d_d02_p00` (Example 23.24): solutions for potential

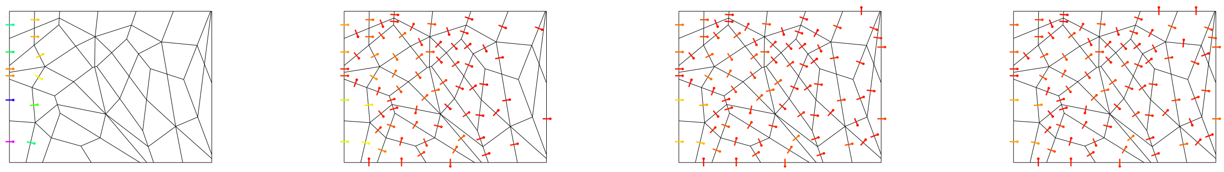
Example 23.25. Consider the transient continuous heat transport problem (Formulation 21.3, Formulation 21.6, Formulation 21.9) with input data `2d_d02_p01` in the nomenclature of the C codebase.
Concretely, $X = [0, 20] \times [0, 15]$, $\tilde{\pi} \equiv 4$, $\tilde{\kappa} \equiv 6$, $u_0(x, y) = \begin{cases} 100, & x = 20 \\ 0, & x < 20 \end{cases}$, $f \equiv 0$, $G_D = \{0, 20\} \times [0, 15]$, $G_N = [0, 20] \times \{0, 15\}$, $g_D(x, y) = \begin{cases} 100, & x = 20 \\ 0, & x = 0 \end{cases}$, $g_N \equiv 0$.
This problem has the following exact solution in steady-state:

$$u(x, y) = 5x, \tag{23.29a}$$
$$q(x, y) = -30\, dy. \tag{23.29b}$$

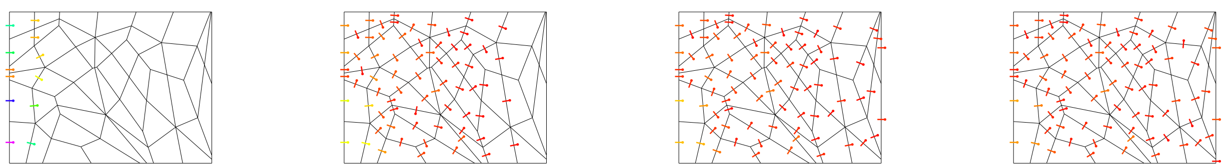
For this problem I use a mesh M generated by Neper with Forman subdivision K. Its potential and flow rate on K consisting of the 3 discussed cochain methods are shown on Figure 62 and Figure 63.
At the moment there are problems with the primal strong method!!!



(a) Primal strong, 1 (b) Primal strong, 334 (c) Primal strong, 667 (d) Primal strong, 1001

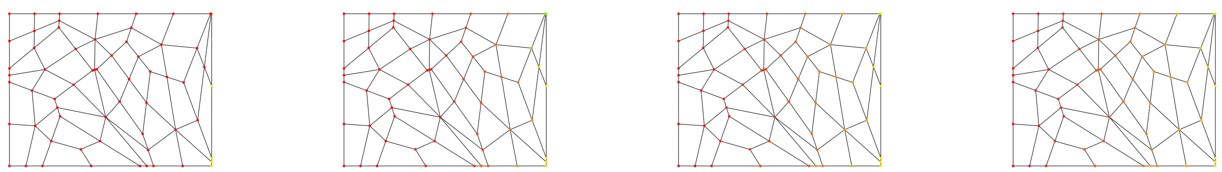


(e) Primal weak, 1 (f) Primal weak, 334 (g) Primal weak, 667 (h) Primal weak, 1001

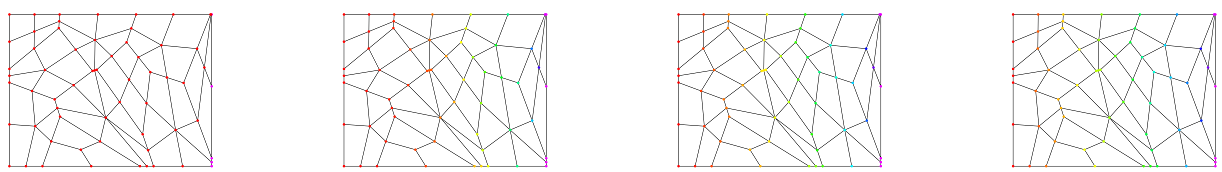


(i) Mixed weak, 1 (j) Mixed weak, 334 (k) Mixed weak, 667 (l) Mixed weak, 1001

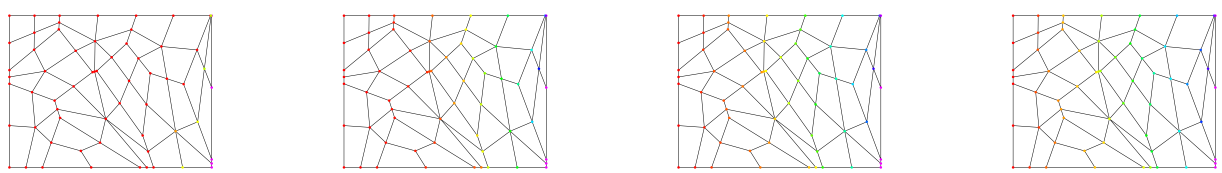
Figure 61: `diffusion/transient/continuous_2d_d02_p00` (Example 23.24): solutions for flow rate



(a) Primal strong, 1 (b) Primal strong, 334 (c) Primal strong, 667 (d) Primal strong, 1001

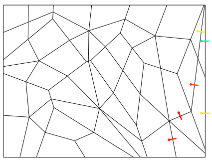


(e) Primal weak, 1 (f) Primal weak, 334 (g) Primal weak, 667 (h) Primal weak, 1001

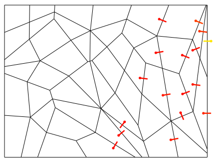


(i) Mixed weak, 1 (j) Mixed weak, 334 (k) Mixed weak, 667 (l) Mixed weak, 1001

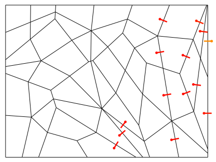
Figure 62: `diffusion/transient/continuous_2d_d02_p01` (Example 23.25): solutions for potential



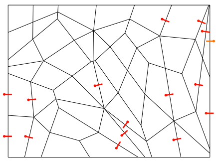
(a) Primal strong, 1



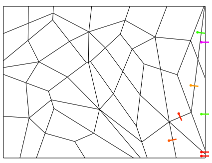
(b) Primal strong, 334



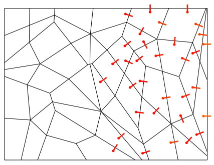
(c) Primal strong, 667



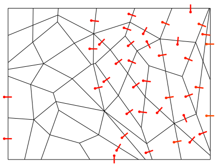
(d) Primal strong, 1001



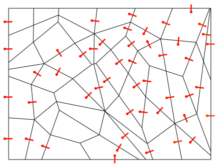
(e) Primal weak, 1



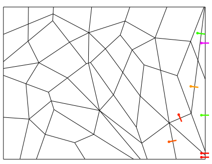
(f) Primal weak, 334



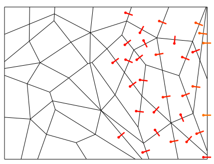
(g) Primal weak, 667



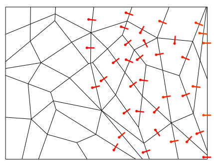
(h) Primal weak, 1001



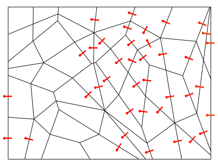
(i) Mixed weak, 1



(j) Mixed weak, 334



(k) Mixed weak, 667



(l) Mixed weak, 1001

Figure 63: `diffusion/transient/continuous_2d_d02_p01` (Example 23.25): solutions for flow rate

Example 23.26. Consider the transient continuous heat transport problem (Formulation 21.3, Formulation 21.6, Formulation 21.9) with input data `2d_d03_p00` in the nomenclature of the C codebase.
Concretely, $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, $\tilde{\pi} \equiv 4$, $\tilde{\kappa} \equiv 1$, $u_0(x, y) = 2 - (x^2 + y^2)$, $f \equiv -4\,dx \wedge dy$, $G_D = \partial X$, $G_N = \emptyset$, $g_D \equiv 1$.
This problem has the following exact solution in steady-state:

$$u(x, y) = x^2 + y^2, \tag{23.30a}$$

$$q(x, y) = 2y\,dx - 2x\,dy. \tag{23.30b}$$

Consider a mesh M for X consisting of n_a rays and n_d disks with Forman subdivision K. Its potential and flow rate on K consisting of the 3 discussed cochain methods for $(n_a, n_d) = (4, 3)$ are shown on Figure 64 and Figure 65.

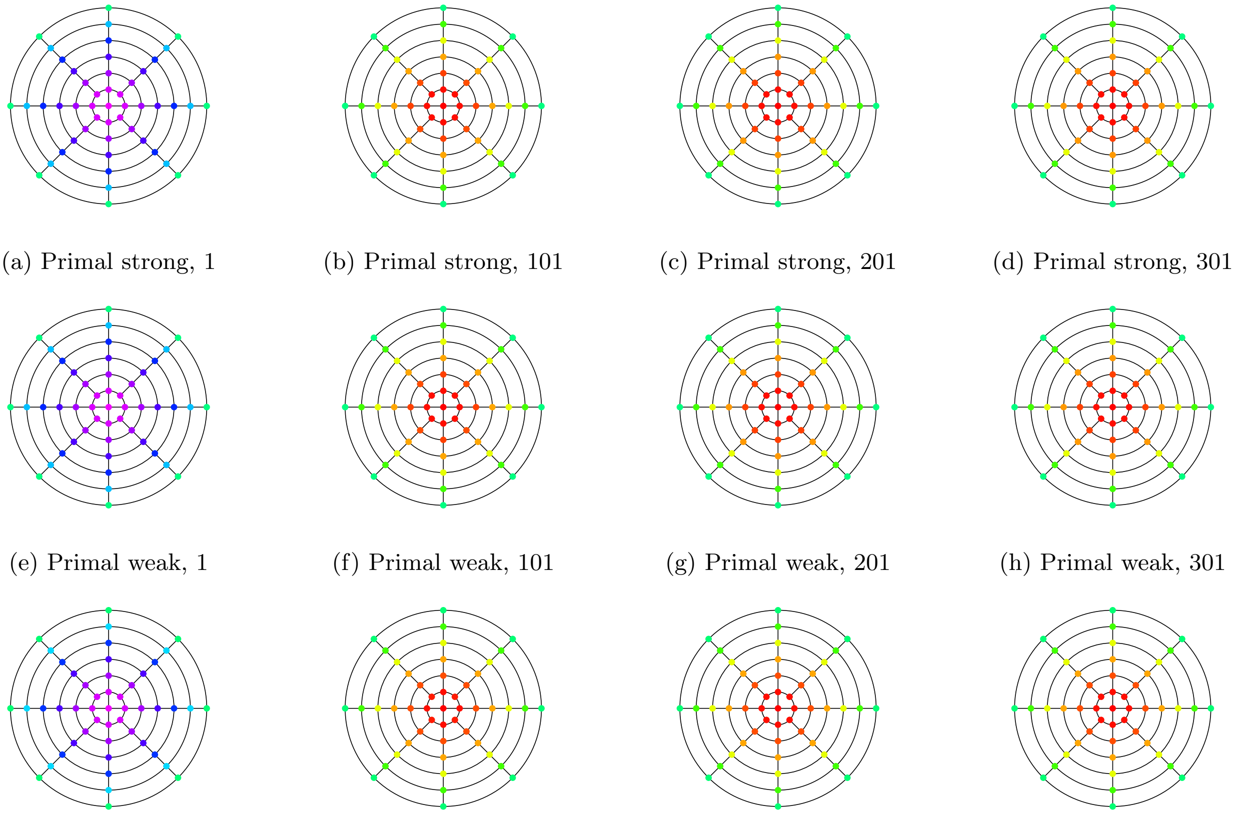
Example 23.27. Consider the transient continuous heat transport problem (Formulation 21.3, Formulation 21.6, Formulation 21.9) with input data `2d_d03_p01` in the nomenclature of the C codebase.
Concretely, $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$, $\tilde{\pi} \equiv 4$, $\tilde{\kappa} \equiv 1$, $u_0(x, y) = 2 - (x^2 + y^2)$, $f \equiv -4\,dx \wedge dy$, $G_D = \{(x, y) \in \partial X \mid x \geq 0\}$, $G_N = \{(x, y) \in \partial X \mid x \leq 0\}$, $g_D \equiv 1$, $g_N(t) = -2\,dt$ (with respect to the $(x, y) = (\cos(t), \sin(t))$ coordinates).
This problem has the following exact solution in steady-state:

$$u(x, y) = x^2 + y^2, \tag{23.31a}$$

$$q(x, y) = 2y\,dx - 2x\,dy. \tag{23.31b}$$

Consider a mesh M for X consisting of n_a rays and n_d disks with Forman subdivision K. Its potential and flow rate on K consisting of the 3 discussed cochain methods for $(n_a, n_d) = (4, 3)$ are shown on Figure 66 and Figure 67.
At the moment there are problems with the primal strong method!!!

24 Continuous electromagnetism

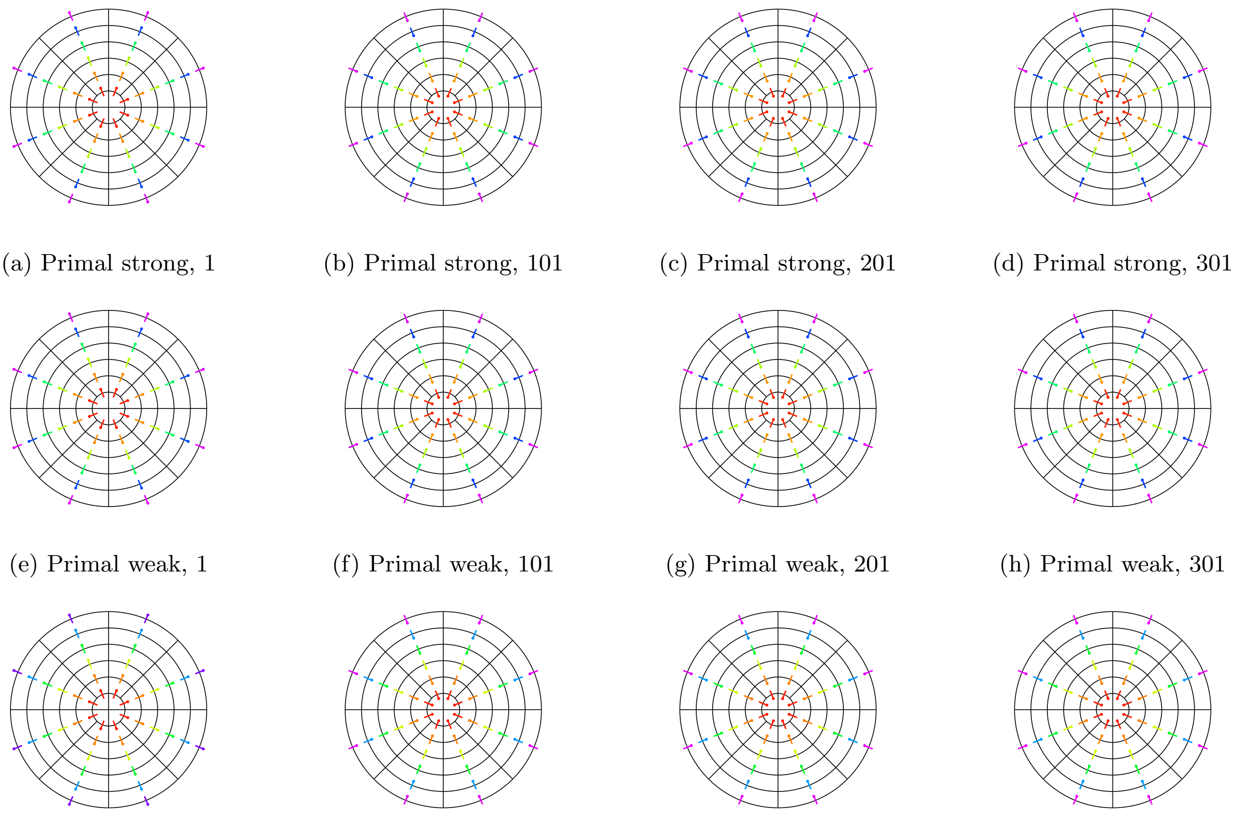


(a) Primal strong, 1 (b) Primal strong, 101 (c) Primal strong, 201 (d) Primal strong, 301

(e) Primal weak, 1 (f) Primal weak, 101 (g) Primal weak, 201 (h) Primal weak, 301

(i) Mixed weak, 1 (j) Mixed weak, 101 (k) Mixed weak, 201 (l) Mixed weak, 301

Figure 64: `diffusion/transient/continuous_2d_d03_p00` (Example 23.26): solutions for potential

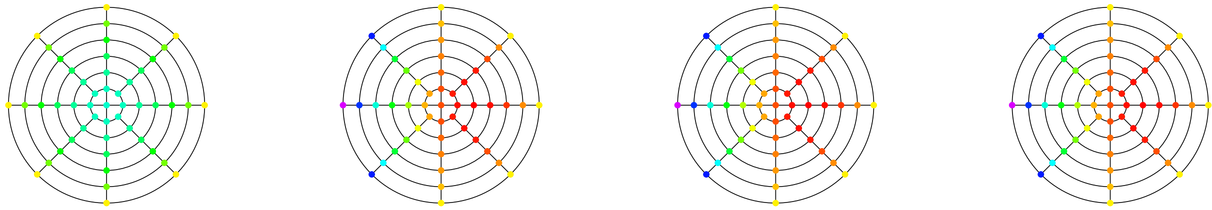


(a) Primal strong, 1 (b) Primal strong, 101 (c) Primal strong, 201 (d) Primal strong, 301

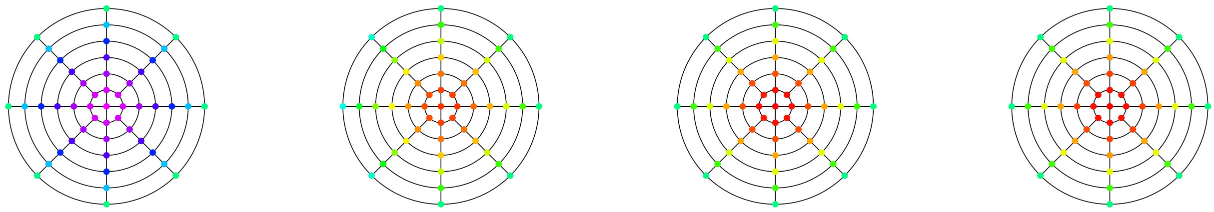
(e) Primal weak, 1 (f) Primal weak, 101 (g) Primal weak, 201 (h) Primal weak, 301

(i) Mixed weak, 1 (j) Mixed weak, 101 (k) Mixed weak, 201 (l) Mixed weak, 301

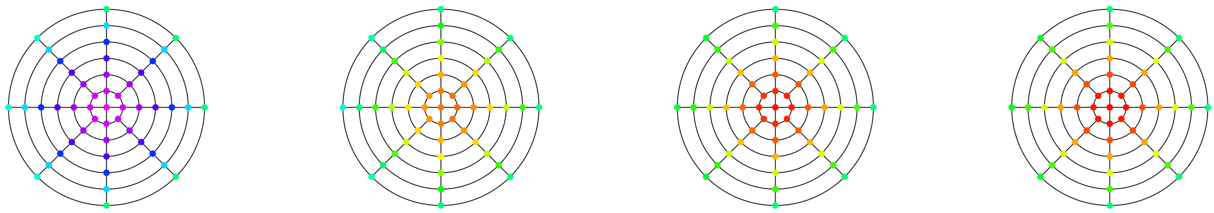
Figure 65: `diffusion/transient/continuous_2d_d03_p00` (Example 23.26): solutions for flow rate



(a) Primal strong, 1 (b) Primal strong, 101 (c) Primal strong, 201 (d) Primal strong, 301

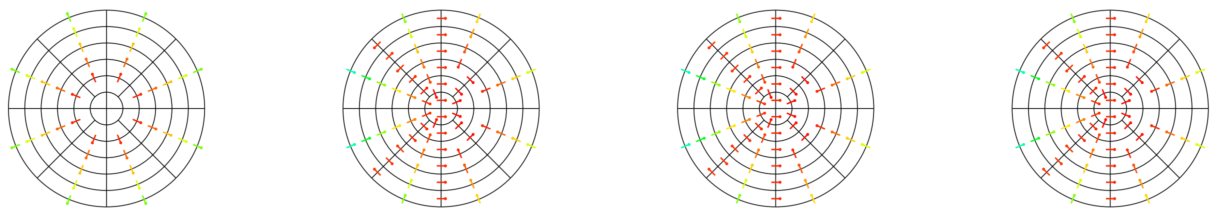


(e) Primal weak, 1 (f) Primal weak, 101 (g) Primal weak, 201 (h) Primal weak, 301

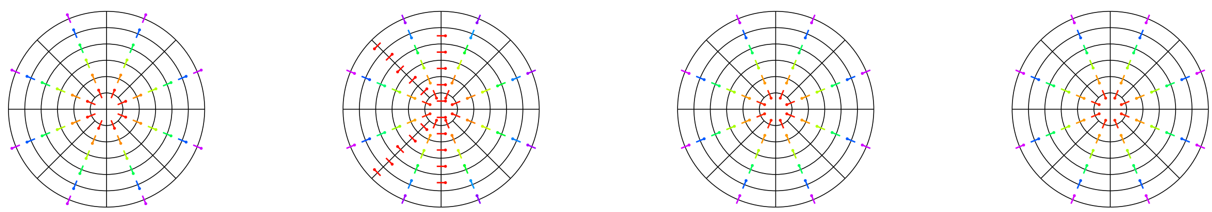


(i) Mixed weak, 1 (j) Mixed weak, 101 (k) Mixed weak, 201 (l) Mixed weak, 301

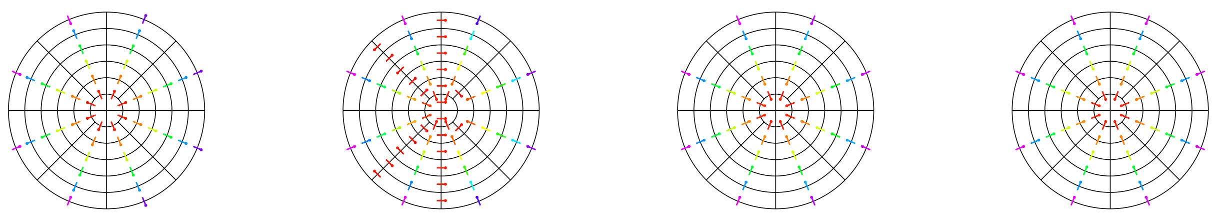
Figure 66: `diffusion/transient/continuous_2d_d03_p01` (Example 23.27): solutions for potential



(a) Primal strong, 1 (b) Primal strong, 101 (c) Primal strong, 201 (d) Primal strong, 301



(e) Primal weak, 1 (f) Primal weak, 101 (g) Primal weak, 201 (h) Primal weak, 301



(i) Mixed weak, 1 (j) Mixed weak, 101 (k) Mixed weak, 201 (l) Mixed weak, 301

Figure 67: `diffusion/transient/continuous_2d_d03_p01` (Example 23.27): solutions for flow rate

Discussion 24.1. The quantities participating in the classical electromagnetism are given in Table 1. The Maxwell's equations and the Poynting's theorem are given in Table 2. The linear constitutive laws in the macroscopic formulation are given in Table 3. (Note that $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$ in the microscopic formulation. Conductivity is not used there since J is given.)

Table 1: Quantities in electromagnetism with forms

Quantity	Variable	Spatial domain	Definition	Dimension
Electric charge	Q	$\Omega^3 M$	given	C
Electric current	J	$\Omega^2 M$	given (or via constitutive law)	$\mathsf{T}^{-1}\mathsf{C}$
Electric potential	φ	$\Omega^0 M$	unknown (gauge freedom)	$\mathsf{ML}^2\mathsf{T}^{-2}\mathsf{C}^{-1}$
Magnetic potential	A	$\Omega^1 M$	unknown (gauge freedom)	$\mathsf{ML}^2\mathsf{T}^{-1}\mathsf{C}^{-1}$
Electric field	E	$\Omega^1 M$	$-d\varphi - \frac{\partial A}{\partial t}$	$\mathsf{ML}^2\mathsf{T}^{-2}\mathsf{C}^{-1}$
Magnetic field	B	$\Omega^2 M$	$d_1 A$	$\mathsf{ML}^2\mathsf{T}^{-1}\mathsf{C}^{-1}$
Electric displacement	D	$\Omega^2 M$	via constitutive law	C
Magnetisation	H	$\Omega^1 M$	via constitutive law	$\mathsf{T}^{-1}\mathsf{C}$
Poynting form	S	$\Omega^2 M$	$E \wedge H$	$\mathsf{ML}^2\mathsf{T}^{-3}$
Electric energy form	$u_{\mathcal{E}}$	$\Omega^3 M$	$(E \wedge D)/2$	$\mathsf{ML}^2\mathsf{T}^{-2}$
Magnetic energy form	$u_{\mathcal{M}}$	$\Omega^3 M$	$(B \wedge H)/2$	$\mathsf{ML}^2\mathsf{T}^{-2}$
Electromagnetic energy form	u	$\Omega^3 M$	$u_{\mathcal{E}} + u_{\mathcal{M}}$	$\mathsf{ML}^2\mathsf{T}^{-2}$
Lorentz force form	F	$\Omega^2 M$	$\star_1(\star_3 Q \wedge E) + \star_2 J \wedge \star_2 B$	MT^{-2}
Permittivity	ε	$\Omega^2 M \to \Omega^2 M$	material parameter	$\mathsf{M}^{-1}\mathsf{L}^{-3}\mathsf{T}^2\mathsf{C}^2$
Permeability	μ	$\Omega^2 M \to \Omega^2 M$	material parameter	MLC^{-2}
Conductivity	σ	$\Omega^2 M \to \Omega^2 M$	material parameter	$\mathsf{M}^{-1}\mathsf{L}^{-3}\mathsf{TC}^2$

Table 2: Laws of electromagnetism with forms

Name	Equation	Domain	Dimension
Gauss's law for electricity	$d_2 D = Q$	$\Omega^3 M$	C
Gauss's law for magnetism	$d_2 B = 0$	$\Omega^3 M$	$\mathsf{ML}^2\mathsf{T}^{-1}\mathsf{C}^{-1}$
Faradey's law of induction	$\frac{\partial B}{\partial t} = -d_1 E$	$\Omega^2 M$	$\mathsf{ML}^2\mathsf{T}^{-2}\mathsf{C}^{-1}$
Ampere's circuital law	$\frac{\partial D}{\partial t} = d_1 H - J$	$\Omega^2 M$	$\mathsf{T}^{-1}\mathsf{C}$
Conservation of electric charge	$\frac{\partial Q}{\partial t} = -d_2 J$	$\Omega^3 M$	$\mathsf{T}^{-1}\mathsf{C}$
Poynting's theorem	$\frac{\partial u}{\partial t} = -d_2 S - E \wedge J$	$\Omega^3 M$	$\mathsf{ML}^2\mathsf{T}^{-3}$

Table 3: Linear constitutive relations in elecromagnetism

Name	Equation	Domain	Dimension
Polarization relation	$D = \varepsilon \star_1 E$	$\Omega^2 M$	C
Magnetization relation	$B = \mu \star_1 H$	$\Omega^2 M$	$\mathsf{ML}^2\mathsf{T}^{-1}\mathsf{C}^{-1}$
Ohm's relation	$J = \sigma \star_1 E$	$\Omega^2 M$	$\mathsf{T}^{-1}\mathsf{C}$

Discussion 24.2. Let M be a space domain manifold, I be an interval. In the split space and time approach quantities in electromagnetism are represented via bundle-vlued functions $f \in \mathcal{F}(I, \Omega^\bullet M)$. We will instead formulate the laws of electromagnetism in terms of $\Omega^\bullet(I \times M)$ (or more generally, a 4-dimensional spacetime that is not

the Cartesian product of space and time). For any $p \in \mathbb{N}$ we will make use of the following isomorphism:

$$C^\infty(I, \Omega^p M) \oplus C^\infty(I, \Omega^{p-1} M) \simeq \Omega^p(I \times M) \tag{24.1}$$

realised by the following map:

$$\omega \mapsto \begin{cases} f\pi_M^* \eta, & \omega = f\eta \in C^\infty(I, \Omega^p M) \\ f\pi_M^* \eta \wedge dt, & \omega = f\eta \in C^\infty(I, \Omega^{p-1} M). \end{cases} \tag{24.2}$$

We form the following pairs and their spacetime versions on Table 4. Consequently, the laws are given on Table 5. Note that the spacetime formulation is manifestly covariant in the category of Lorentzian 4-manifolds (pseudo-Riemannian manifolds with $(1,3)$ signature of the metric).

Table 4: Quantities in spacetime electromagnetism

Quantity	Variable	Domain	Definition	Dimension
Spacetime charge	$\mathcal{Q}$	$\Omega^3(I \times M)$	$Q + J \wedge dt$	C
Electromagnetic potential	$\mathcal{A}$	$\Omega^1(I \times M)$	$\varphi\, dt + A$	$\mathsf{ML^2T^{-1}C^{-1}}$
Electromagnetic field	$\mathcal{F}$	$\Omega^2(I \times M)$	$B + E \wedge dt$	$\mathsf{ML^2T^{-1}C^{-1}}$
Electromagnetic displacement	$\mathcal{D}$	$\Omega^2(I \times M)$	$D + H \wedge dt$	C

Table 5: Laws of spacetime electromagnetism

Name	Equation	Domain	Dimension
Electromagnetic field is closed	$d_2 \mathcal{F} = 0$	$\Omega^3(I \times M)$	$\mathsf{ML^2T^{-1}C^{-1}}$
Conservation of electric charge: strong	$d_2 \mathcal{D} = \mathcal{Q}$	$\Omega^3(I \times M)$	C
Conservation of electric charge: weak	$d_3 \mathcal{Q} = 0$	$\Omega^4(I \times M)$	C
Electromagnetic form is exact	$d_1 \mathcal{A} = \mathcal{F}$	$\Omega^2(I \times M)$	$\mathsf{ML^2T^{-1}C^{-1}}$
Constitutive law	$\mathcal{F} = \varepsilon \star_2 \mathcal{D}$	$\Omega^2(I \times M)$	$\mathsf{ML^2T^{-1}C^{-1}}$

25 Discrete elasticity

Discussion 25.1 (Discrete elasticity)**.** Let M be a mesh of dimension 3, K be the Forman subdivision of M. Let L and F denote length and force measures respectively.
Discrete displacement is represented by

$$\eta^1[L^2] \in C^1 K. \tag{25.1}$$

Displacement gradient is represented by

$$\epsilon^0[1] \in C^0 K, \;\; \omega^2[L^2] \in C^2 K. \tag{25.2}$$

Stress (force) is represented by

$$\tau^0[F] \in C^0 K, \;\; \tau^2[FL^2] \in C^2 K. \tag{25.3}$$

Body force is represented by

$$\mathfrak{b}^1[F] \in C^1 K. \tag{25.4}$$

Let $\lambda, \mu[F] \in \mathbb{R}$ be the Lamé parameters. Our model is the following.

$$\epsilon^0 = \delta_1^\star \eta^1 \quad \text{(volumetric displacement gradient),} \tag{25.5a}$$

$$\omega^2 = \delta_1 \eta^1 \quad \text{(deviatoric displacement gradient),} \tag{25.5b}$$

$$\tau^0 = \lambda \epsilon^0 \quad \text{(hydrostatic force),} \tag{25.5c}$$

$$\tau^2 = \mu \omega^2 \quad \text{(deviatoric force),} \tag{25.5d}$$

$$\delta_0 \tau^0 + \delta_2^\star \tau^2 + \mathfrak{b}^1 = 0 \quad \text{(conservation of linear momentum).} \tag{25.5e}$$

Example 25.2. Consider the problem of a twist of a cylindrical bar described in Section 9.1 of (Hadjesfandiari 2011). Let θ be the constant angle of twist per unit length, λ and μ be the Lame parameters. Let $\mathbf{u}$ be the displacement vector, $\boldsymbol{\epsilon}$ be the strain tensor, $\boldsymbol{\omega}$ be the rotation tensor, $\boldsymbol{\sigma}$ be the stress tensor. Then at any point $x = (x_1, x_2, x_3)$ we have:

$$\mathbf{u} = \begin{pmatrix} -\theta x_2 x_3 \\ \theta x_1 x_3 \\ 0 \end{pmatrix}, \ \boldsymbol{\epsilon} = \frac{\theta}{2}\begin{pmatrix} 0 & 0 & -x_2 \\ 0 & 0 & x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}, \ \boldsymbol{\omega} = \frac{\theta}{2}\begin{pmatrix} 0 & -2x_3 & -x_2 \\ 2x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}, \ \boldsymbol{\sigma} = \mu\theta \begin{pmatrix} 0 & 0 & -x_2 \\ 0 & 0 & x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}. \quad (25.6)$$

Note that the skew-symmetric matrix $\boldsymbol{\omega}$ corresponds to the vector

$$\boldsymbol{\omega} \mapsto \frac{\theta}{2}(-x_1, -x_2, 2x_3)^T. \quad (25.7)$$

Let $h \in \mathbb{R}^+$ and consider a 3D regular grid K of size h. For integers i, j, k, nodes in K have coordinates

$$\mathbf{x}_{(i,j,k)} := (ih, jh, kh). \quad (25.8)$$

Nodes in K will be denoted by $\mathcal{N}_{(i,j,k)}$.
There are three type of edges in K (parallel to the 3 axes) constructed as follows. Let $p \in \{1,2,3\}$ and e_p be the p-th unit vector. Denote by $\mathcal{E}^{(p)}_{(i,j,k)}$ the edge starting at $\mathcal{N}_{(i,j,k)}$ and ending at $\mathcal{N}_{(i,j,k)+e_p}$. In particular, the oriented boundary of $\mathcal{E}^{(1)}_{(i,j,k)}$ is

$$\partial_1 \mathcal{E}^{(1)}_{(i,j,k)} = -\mathcal{N}_{(i,j,k)} + \mathcal{N}_{(i+1,j,k)}. \quad (25.9)$$

Similar computation holds for $p = 2$ and $p = 3$.
For $p, q \in \{1,2,3\}$, $p < q$ faces in K are denoted by $\mathcal{F}^{(p,q)}_{(i,j,k)}$ and represent squares starting at $\mathcal{N}_{(i,j,k)}$ with basis vectors going in directions e_p and e_q. Also use the identification of cochains

$$\mathcal{F}^{(q,p)}_{(i,j,k)} := -\mathcal{F}^{(p,q)}_{(i,j,k)}. \quad (25.10)$$

For instance, the oriented boundary of $\mathcal{F}^{(1,2}_{(i,j,k)}$ is

$$\partial_2 \mathcal{F}^{(1,2)}_{(i,j,k)} = -\mathcal{E}^{(1)}_{(i,j+1,k)} + \mathcal{E}^{(1)}_{(i,j,k)} + \mathcal{E}^{(2)}_{(i+1,j,k)} - \mathcal{E}^{(2)}_{(i,j,k)}. \quad (25.11)$$

We will work with the approximation $\eta^1 := \underline{u}$.
Let $p \in \{1,2,3\}$. Then for $\epsilon^0 := \delta_1^\star \eta^1$, using the fact that

$$\eta^1 \mathcal{E}^{(p)}_{(i,j,k)+e_p} = \eta^1 \mathcal{E}^{(p)}_{(i,j,k)}, \quad (25.12)$$

we calculate:

$$\epsilon^0 \mathcal{N}_{(i,j,k)} = \frac{1}{h^2}\sum_{p=1}^{3}(\eta^1 \mathcal{E}^{(p)}_{(i,j,k)} - \eta^1 \mathcal{E}^{(p)}_{(i,j,k)+e_p}) = 0. \quad (25.13)$$

Hence,

$$\tau^0 = \lambda \epsilon^0 = 0. \quad (25.14)$$

Using the computation from Equation (18.11) in each direction, we get

$$\eta^1 \mathcal{E}^{(1)}_{(i,j,k)} = \frac{h}{2}\theta(-x_2x_3 - x_2x_3) = -\theta h x_2 x_3 = -\theta jkh^3, \quad (25.15a)$$
$$\eta^1 \mathcal{E}^{(2)}_{(i,j,k)} = \frac{h}{2}\theta(x_1x_3 + x_1x_3) = \theta h x_1 x_3 = \theta ikh^3, \quad (25.15b)$$
$$\eta^1 \mathcal{E}^{(3)}_{(i,j,k)} = \frac{h}{2}\theta(0 + 0) = 0. \quad (25.15c)$$

For $\omega^2 := \delta_1 \eta^1$, using that $\epsilon(c_2) = \eta^1(\partial c_2)$, we get:

$$\omega^2 \mathcal{F}^{(2,3)}_{(i,j,k)} = (-\eta^1 \mathcal{E}^{(2)}_{(i,j,k+1)} + \eta^1 \mathcal{E}^{(2)}_{(i,j,k)}) + (\eta^1 \mathcal{E}^{(3)}_{(i,j+1,k)} - \eta^1 \mathcal{E}^{(3)}_{(i,j,k)}) = -\theta ih^3 + 0 \quad = -\theta ih^3, \quad (25.16a)$$
$$\omega^2 \mathcal{F}^{(3,1)}_{(i,j,k)} = (-\eta^1 \mathcal{E}^{(3)}_{(i+1,j,k)} + \eta^1 \mathcal{E}^{(3)}_{(i,j,k)}) + (\eta^1 \mathcal{E}^{(1)}_{(i,j,k+1)} + \eta^1 \mathcal{E}^{(1)}_{(i,j,k)}) = 0 - \theta jh^3 \quad = -\theta jh^3, \quad (25.16b)$$
$$\omega^2 \mathcal{F}^{(1,2)}_{(i,j,k)} = (-\eta^1 \mathcal{E}^{(1)}_{(i,j+1,k)} + \eta^1 \mathcal{E}^{(1)}_{(i,j,k)}) + (\eta^1 \mathcal{E}^{(2)}_{(i+1,j,k)} - \eta^1 \mathcal{E}^{(2)}_{(i,j,k)}) = \theta kh^3 + \theta kh^3 = 2\theta kh^3. \quad (25.16c)$$

We see the clear correspondence (by a factor of $2h^2$) to the "flattened" version of $\boldsymbol{\omega}$, Equation (25.7).
We have $\tau^2 = \mu\omega^2$ and hence $\delta_2^\star\tau^2 = \mu\delta_2^\star\omega^2$. Then

$$(\delta_2^\star\tau^2)\mathcal{E}^{(1)}_{(i,j,k)} = \frac{\mu}{h^2}((\omega^2\mathcal{F}^{(1,2)}_{(i,j,k)} - \omega^2\mathcal{F}^{(1,2)}_{(i,j-1,k)}) - (\omega^2\mathcal{F}^{(1,3)}_{(i,j,k)} - \omega^2\mathcal{F}^{(1,3)}_{(i,j,k-1)})) = \frac{\mu}{h^2}(0-0) = 0, \tag{25.17a}$$

$$(\delta_2^\star\tau^2)\mathcal{E}^{(2)}_{(i,j,k)} = \frac{\mu}{h^2}((\omega^2\mathcal{F}^{(1,2)}_{(i-1,j,k)} - \omega^2\mathcal{F}^{(1,2)}_{(i,j,k)}) - (\omega^2\mathcal{F}^{(2,3)}_{(i,j,k)} - \omega^2\mathcal{F}^{(2,3)}_{(i,j,k-1)})) = \frac{\mu}{h^2}(0-0) = 0, \tag{25.17b}$$

$$(\delta_2^\star\tau^2)\mathcal{E}^{(3)}_{(i,j,k)} = \frac{\mu}{h^2}((\omega^2\mathcal{F}^{(1,3)}_{(i-1,j,k)} - \omega^2\mathcal{F}^{(1,3)}_{(i,j,k)}) - (\omega^2\mathcal{F}^{(2,3)}_{(i,j-1,k)} - \omega^2\mathcal{F}^{(2,3)}_{(i,j,k)})) = \frac{\mu}{h^2}(0-0) = 0. \tag{25.17c}$$

Hence, $\delta_2^\star\tau^2 = 0$. With zero body force $\mathfrak{b}^1$, we get:

$$\delta_0\tau^0 + \delta_2^\star\tau^2 + \mathfrak{b}^1 = 0 + 0 + 0 = 0. \tag{25.18}$$